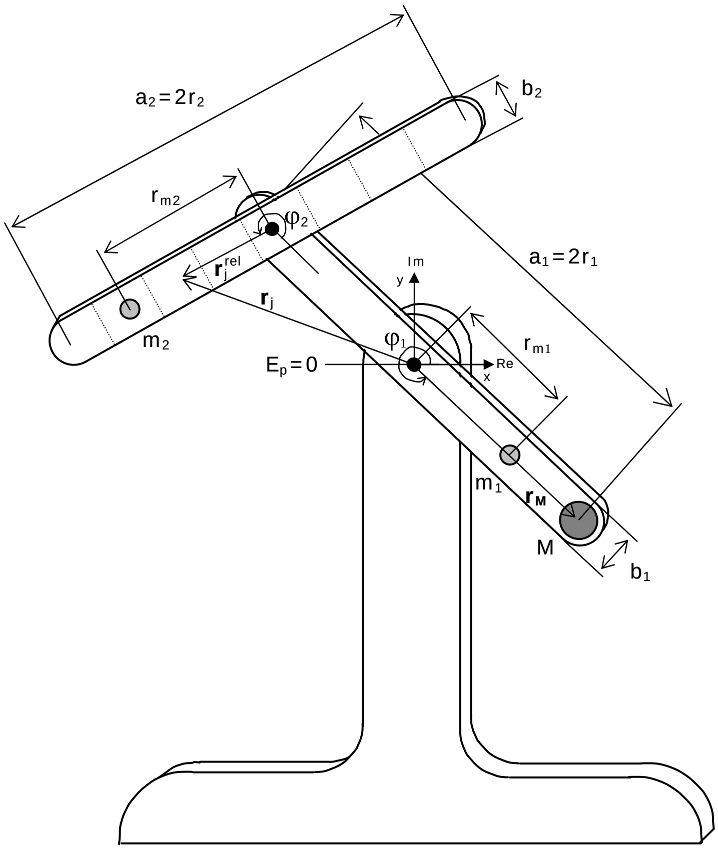

Position and velocity

The pendulum has two arms equipped with weights m_1 and m_2. The inner arm has a counterweight M. Vectors are given in the inertial system with fixed axes and centre at the inner axis. The system has two degrees of freedom described by the angles φ_1 and φ_2. φ_2 is the angular position measured from the inner arm. The position of the outer arm in the fixed system is given by $\varphi_1 + \varphi_2$.

Position: $\mathbf{r} = (x, y) \sim x + iy = re^{i\varphi}$ Velocity: $\mathbf{v} = \dot{\mathbf{r}} = \dot{r}e^{i\varphi} + ire^{i\varphi}\dot{\varphi}$

Arm1

weight m_1: $\mathbf{r}_{m1} = r_{m1}e^{i\varphi 1}$ $\mathbf{v}_{m1} = \dot{\mathbf{r}}_{m1} = ir_{m1}e^{i\varphi 1}\dot{\varphi}_1$

weight M: $\mathbf{r}_M = r_1 e^{i\varphi 1}$ $\mathbf{v}_M = \dot{\mathbf{r}}_M = ir_1 e^{i\varphi 1}\dot{\varphi}_1$

Arm2

weight m_2: $\mathbf{r}_{m2} = -\mathbf{r}_M + \mathbf{r}_{m2}^{rel} = -r_1 e^{i\varphi 1} + r_{m2}e^{i(\varphi 1 + \varphi 2)}$

$$\mathbf{v}_{m2} = -\dot{\mathbf{r}}_M + \dot{\mathbf{r}}_{m2}^{rel} = -ir_1 e^{i\varphi 1}\dot{\varphi}_1 + ir_{m2}e^{i(\varphi 1 + \varphi 2)}(\dot{\varphi}_1 + \dot{\varphi}_2)$$

segment j: $\mathbf{r}_j = -\mathbf{r}_M + \mathbf{r}_j^{rel} = -r_1 e^{i\varphi 1} + r_j e^{i(\varphi 1 + \varphi 2)}$

$$\mathbf{v}_j = -\dot{\mathbf{r}}_M + \dot{\mathbf{r}}_j^{rel} = -ir_1 e^{i\varphi 1}\dot{\varphi}_1 + ir_j e^{i(\varphi 1 + \varphi 2)}(\dot{\varphi}_1 + \dot{\varphi}_2)$$

Kinetic energy $E_k = E_{1k} + E_{2k}$

Arm1:

$$E_{1k} = \frac{I_{arm1}\cdot\dot{\varphi}_1^{\,2}}{2} + \frac{I_M\cdot\dot{\varphi}_1^{\,2}}{2} + \frac{I_{m1}\cdot\dot{\varphi}_1^{\,2}}{2}$$

Arm2:

$$E_{2k} = \frac{m_2\cdot v_{m2}^{\,2}}{2} + \sum_j \frac{m_j\cdot v_j^{\,2}}{2}$$

Moment of inertia

$$I_{arm1} = m_{arm1}\cdot\frac{a_1^{\,2}+b_1^{\,2}}{12}$$

$$I_M = M\cdot r_1^2$$

$$I_{m1} = m_1\cdot r_{m1}^{\,2}$$

$$I_{arm2} = \sum_j m_j r_j^{\,2} = m_{arm2}\cdot\frac{a_2^{\,2}+b_2^{\,2}}{12}$$

$$I_{m2} = m_2\cdot r_{m2}^{\,2}$$

$$\begin{aligned} v_{m2}^{\,2} = \boldsymbol{v}_{m2}\cdot\bar{\boldsymbol{v}}_{m2} &= (-ir_1e^{i\varphi_1}\dot{\varphi}_1 + ir_{m2}e^{i(\varphi_1+\varphi_2)}(\dot{\varphi}_1+\dot{\varphi}_2))\cdot(ir_1e^{-i\varphi_1}\dot{\varphi}_1 - ir_{m2}e^{-i(\varphi_1+\varphi_2)}(\dot{\varphi}_1+\dot{\varphi}_2)) = \\ &= r_1^2\dot{\varphi}_1^{\,2} + r_{m2}^{\,2}(\dot{\varphi}_1+\dot{\varphi}_2)^2 - 2r_1r_{m2}\dot{\varphi}_1(\dot{\varphi}_1+\dot{\varphi}_2)\cos\varphi_2 = \\ &= (r_1^2 + r_{m2}^{\,2} - 2r_1r_{m2}\cos\varphi_2)\dot{\varphi}_1^{\,2} + r_{m2}^{\,2}\dot{\varphi}_2^{\,2} + (2r_{m2}^{\,2} - 2r_1r_{m2}\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2 \end{aligned}$$

$$v_j^{\,2} = (r_1^2 + r_j^{\,2} - 2r_1r_j\cos\varphi_2)\dot{\varphi}_1^{\,2} + r_j^{\,2}\dot{\varphi}_2^{\,2} + (2r_j^{\,2} - 2r_1r_j\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2$$

$$\begin{aligned} E_k = E_{1k} + E_{2k} &= \frac{I_{arm1}+I_M+I_{m1}}{2}\dot{\varphi}_1^{\,2} + \\ &\frac{m_2(r_1^2 + r_{m2}^{\,2} - 2r_1r_{m2}\cos\varphi_2)}{2}\dot{\varphi}_1^{\,2} + \frac{m_2r_{m2}^{\,2}}{2}\dot{\varphi}_2^{\,2} + \frac{m_2(2r_{m2}^{\,2} - 2r_1r_{m2}\cos\varphi_2)}{2}\dot{\varphi}_1\dot{\varphi}_2 + \\ &\sum_j \frac{m_j}{2}\Big((r_1^2 + r_j^{\,2} - 2r_1r_j\cos\varphi_2)\dot{\varphi}_1^{\,2} + r_j^{\,2}\dot{\varphi}_2^{\,2} + (2r_j^{\,2} - 2r_1r_j\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2\Big) = \\ &\left(\frac{I_{arm1}+I_M+I_{m1}+m_2r_1^2+I_{m2}-2m_2r_1r_{m2}\cos\varphi_2+m_{arm2}r_1^2+I_{arm2}}{2}\right)\dot{\varphi}_1^{\,2} + \\ &\left(\frac{I_{m2}+I_{arm2}}{2}\right)\dot{\varphi}_2^{\,2} + \left(I_{m2} - m_2r_1r_{m2}\cos\varphi_2 + I_{arm2}\right)\dot{\varphi}_1\dot{\varphi}_2 \end{aligned}$$

$$\left(\sum_j m_j r_j = 0\right)$$

$$A = (I_{arm1} + I_M + I_{m1} + I_{arm2} + I_{m2} + m_2r_1^2 + m_{arm2}r_1^2)/2$$

$$B = m_2r_1r_{m2}$$

$$C = (I_{arm2} + I_{m2})/2$$

$$E_k = (A - B\cos\varphi_2)\dot{\varphi}_1^{\,2} + C\dot{\varphi}_2^{\,2} + (2C - B\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2$$

Potential energy $E_p = E_{1p} + E_{2p}$

Arm1: $E_{1p} = Mgr_1 \sin\varphi_1 + m_1 g r_{m1} \sin\varphi_1$ (zero at inner axis)

Arm2: $E_{2p} = m_{arm2} g \cdot (-r_1 \sin\varphi_1) + m_2 g \cdot (-r_1 \sin\varphi_1 + r_{m2} \sin(\varphi_1 + \varphi_2))$

$$D = (Mr_1 + m_1 r_{m1} - m_{arm2} r_1 - m_2 r_1) g$$
$$E = m_2 r_{m2} g$$

$$E_p = D \sin\varphi_1 + E \sin(\varphi_1 + \varphi_2)$$

Movement

The pendulum motion in the inertial system is given by the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_i}\right) - \frac{\partial L}{\partial \varphi_i} = 0$$

$$L = E_k - E_p = (A - B\cos\varphi_2)\dot{\varphi}_1{}^2 + C\dot{\varphi}_2{}^2 + (2C - B\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2 - D\sin\varphi_1 - E\sin(\varphi_1 + \varphi_2)$$

$$\frac{\partial L}{\partial \varphi_1} = -D\cos\varphi_1 - E\cos(\varphi_1 + \varphi_2)$$
$$\frac{\partial L}{\partial \dot{\varphi}_1} = 2(A - B\cos\varphi_2)\dot{\varphi}_1 + (2C - B\cos\varphi_2)\dot{\varphi}_2$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_1}\right) = 2B\dot{\varphi}_1\dot{\varphi}_2 \sin\varphi_2 + 2(A - B\cos\varphi_2)\ddot{\varphi}_1 + B\dot{\varphi}_2{}^2 \sin\varphi_2 + (2C - B\cos\varphi_2)\ddot{\varphi}_2$$

$$\frac{\partial L}{\partial \varphi_2} = B\dot{\varphi}_1{}^2 \sin\varphi_2 + B\dot{\varphi}_1\dot{\varphi}_2 \sin\varphi_2 - E\cos(\varphi_1 + \varphi_2)$$
$$\frac{\partial L}{\partial \dot{\varphi}_2} = 2C\dot{\varphi}_2 + (2C - B\cos\varphi_2)\dot{\varphi}_1$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_2}\right) = 2C\ddot{\varphi}_2 + B\dot{\varphi}_1\dot{\varphi}_2 \sin\varphi_2 + (2C - B\cos\varphi_2)\ddot{\varphi}_1$$

Friction

Frictional forces that can be derived from a dissipation function $\Im$ can be handled with Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}+\frac{\partial \Im}{\partial \dot{q}_i}=0$$

$$\textit{Frictional force} \quad \mathrm{F}=-\nabla_{\dot{q}}\Im$$

$$\textit{Energy loss due to } \mathrm{F}: \ \frac{dE}{dt}=2\Im$$

Friction of type **F**=-k**v** is derived from $\Im = ½(k_x v_x^2 + k_y v_y^2 + k_z v_z^2)$
Assume that the friction can be derived from $\Im = k_1 E_{1k} + k_2 E_{2k}$

$$\begin{aligned}\Im = {} & k_1 \frac{I_{arm1}+I_M+I_{m1}}{2}\dot{\varphi}_1^{\,2} + \\ & k_2\left(\frac{m_2 r_1^2 + I_{m2} - 2m_2 r_1 r_{m2}\cos\varphi_2 + m_{arm2} r_1^2 + I_{arm2}}{2}\right)\dot{\varphi}_1^{\,2} + \\ & k_2\left(\frac{I_{m2}+I_{arm2}}{2}\right)\dot{\varphi}_2^{\,2} + k_2\left(I_{m2} - m_2 r_1 r_{m2}\cos\varphi_2 + I_{arm2}\right)\dot{\varphi}_1\dot{\varphi}_2\end{aligned}$$

$$\begin{aligned}A_1 &= (I_{arm1}+I_M+I_{m1})/2\\ A_2 &= (I_{arm2}+I_{m2}+m_2 r_1^2 + m_{arm2} r_1^2)/2\\ B &= m_2 r_1 r_{m2}\\ C &= (I_{arm2}+I_{m2})/2\end{aligned}$$

$$\Im = k_1 A_1 \dot{\varphi}_1^{\,2} + k_2 A_2 \dot{\varphi}_1^{\,2} - k_2 B \dot{\varphi}_1^{\,2}\cos\varphi_2 + k_2 C\dot{\varphi}_2^{\,2} + k_2(2C - B\cos\varphi_2)\dot{\varphi}_1\dot{\varphi}_2$$

$$\frac{\partial \Im}{\partial \dot{\varphi}_1} = 2(k_1 A_1 + k_2 A_2 - k_2 B\cos\varphi_2)\dot{\varphi}_1 + k_2(2C - B\cos\varphi_2)\dot{\varphi}_2$$

$$\frac{\partial \Im}{\partial \dot{\varphi}_2} = 2k_2 C\dot{\varphi}_2 + k_2(2C - B\cos\varphi_2)\dot{\varphi}_1$$

Differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_i}\right)=\frac{\partial L}{\partial \varphi_i}-\frac{\partial \Im}{\partial \dot{\varphi}_i}$$

$$\Downarrow$$

$$\begin{cases} 2B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2 + 2(A - B\cos\varphi_2)\ddot{\varphi}_1 + B\dot{\varphi}_2^{\,2}\sin\varphi_2 + (2C - B\cos\varphi_2)\ddot{\varphi}_2 = \\ -D\cos\varphi_1 - E\cos(\varphi_1+\varphi_2) - 2(k_1 A_1 + k_2 A_2 - k_2 B\cos\varphi_2)\dot{\varphi}_1 - k_2(2C - B\cos\varphi_2)\dot{\varphi}_2 \\ \\ 2C\ddot{\varphi}_2 + B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2 + (2C - B\cos\varphi_2)\ddot{\varphi}_1 = \\ B\dot{\varphi}_1^{\,2}\sin\varphi_2 + B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2 - E\cos(\varphi_1+\varphi_2) - 2k_2 C\dot{\varphi}_2 - k_2(2C - B\cos\varphi_2)\dot{\varphi}_1 \end{cases}$$

$$\begin{cases} 2(A-B\cos\varphi_2)\ddot{\varphi}_1+(2C-B\cos\varphi_2)\ddot{\varphi}_2 = \\ -2B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2-B\dot{\varphi}_2{}^2\sin\varphi_2-D\cos\varphi_1-E\cos(\varphi_1+\varphi_2) \\ -2(k_1A_1+k_2A_2-k_2B\cos\varphi_2)\dot{\varphi}_1-k_2(2C-B\cos\varphi_2)\dot{\varphi}_2 \\ \\ (2C-B\cos\varphi_2)\ddot{\varphi}_1+2C\ddot{\varphi}_2 = \\ B\dot{\varphi}_1{}^2\sin\varphi_2-E\cos(\varphi_1+\varphi_2)-2k_2C\dot{\varphi}_2-k_2(2C-B\cos\varphi_2)\dot{\varphi}_1 \end{cases}$$

$(2C-B\cos\varphi_2)\cdot Ekv2-2C\cdot Ekv1$

$$\begin{cases} 2(A-B\cos\varphi_2)\ddot{\varphi}_1+(2C-B\cos\varphi_2)\ddot{\varphi}_2 = \\ -2B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2-B\dot{\varphi}_2{}^2\sin\varphi_2-D\cos\varphi_1-E\cos(\varphi_1+\varphi_2)+ \\ 2(k_2B\cos\varphi_2-k_1A_1-k_2A_2)\dot{\varphi}_1-k_2(2C-B\cos\varphi_2)\dot{\varphi}_2 \\ \\ ((2C-B\cos\varphi_2)^2-4C(A-B\cos\varphi_2))\ddot{\varphi}_1 = \\ 2CD\cos\varphi_1+(4C(k_1A_1+k_2A_2)-k_2(4C^2+B^2\cos^2\varphi_2))\dot{\varphi}_1+ \\ BE\cos\varphi_2\cos(\varphi_1+\varphi_2)+4BC\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2+ \\ (2BC-B^2\cos\varphi_2)\dot{\varphi}_1{}^2\sin\varphi_2+2BC\dot{\varphi}_2{}^2\sin\varphi_2 \end{cases}$$

$$f(\varphi_1,\varphi_2,\dot{\varphi}_1,\dot{\varphi}_2)=$$
$$\{2CD\cos\varphi_1+(4C(k_1A_1+k_2A_2)-k_2(4C^2+B^2\cos^2\varphi_2))\dot{\varphi}_1+BE\cos\varphi_2\cos(\varphi_1+\varphi_2)+4BC\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2+$$
$$(2BC-B^2\cos\varphi_2)\dot{\varphi}_1{}^2\sin\varphi_2+2BC\dot{\varphi}_2{}^2\sin\varphi_2\}\big/\{(2C-B\cos\varphi_2)^2-4C(A-B\cos\varphi_2)\}$$

$$g(\varphi_1,\varphi_2,\dot{\varphi}_1,\dot{\varphi}_2)=$$
$$\{2(B\cos\varphi_2-A)\cdot f(\varphi_1,\varphi_2,\dot{\varphi}_1,\dot{\varphi}_2)-2B\dot{\varphi}_1\dot{\varphi}_2\sin\varphi_2-B\dot{\varphi}_2{}^2\sin\varphi_2-D\cos\varphi_1-E\cos(\varphi_1+\varphi_2)+$$
$$2(k_2B\cos\varphi_2-k_1A_1-k_2A_2)\dot{\varphi}_1-k_2(2C-B\cos\varphi_2)\dot{\varphi}_2\}\big/(2C-B\cos\varphi_2)$$

$$\begin{cases} \ddot{\varphi}_1 = f(\varphi_1,\varphi_2,\dot{\varphi}_1,\dot{\varphi}_2) \\ \ddot{\varphi}_2 = g(\varphi_1,\varphi_2,\dot{\varphi}_1,\dot{\varphi}_2) \end{cases}$$

For the special case of a balanced outer arm $r_{m2}=0$ you get B=0, E=0.

$$\begin{cases} \ddot{\varphi}_1 = ((k_1A_1+k_2A_2-k_2C)\dot{\varphi}_1+\tfrac{1}{2}D\cos\varphi_1)/(C-A) \\ \ddot{\varphi}_2 = (A_1(k_2-k_1)\dot{\varphi}_1+k_2(A-C)\dot{\varphi}_2-\tfrac{1}{2}D\cos\varphi_1)/(C-A) \end{cases}$$

Arm1 moves independently from arm2 but arm2 is affected by arm1.

With no friction, $k_1=0$ and $k_2=0$:

$$\begin{cases} \ddot{\varphi}_1 = \tfrac{1}{2}D\cos\varphi_1/(C-A) \\ \ddot{\varphi}_2 = -\tfrac{1}{2}D\cos\varphi_1/(C-A) \end{cases} \Rightarrow \ddot{\varphi}_1+\ddot{\varphi}_2=0 \Rightarrow \frac{d}{dt}(\varphi_1+\varphi_2)=Konst.$$

Arm2 rotates with constant angular speed in the fixed system.

Numerical analysis

Runge-Kuttas method can be used to get a numerical solution to the differential equations. The equations are transformed to a system of differential equations of the first order.

$$
\begin{matrix} y_1(t)=\varphi_1(t) \\ y_2(t)=\varphi_2(t) \\ y_3(t)=\dot{\varphi}_1(t) \\ y_4(t)=\dot{\varphi}_2(t) \end{matrix}
\quad \rightarrow \quad
\begin{cases} \dot{y}_1 = y_3 & y_1(0)=\varphi_1(0) \\ \dot{y}_2 = y_4 & y_2(0)=\varphi_2(0) \\ \dot{y}_3 = f(y_1,y_2,y_3,y_4) & y_3(0)=\dot{\varphi}_1(0) \\ \dot{y}_4 = g(y_1,y_2,y_3,y_4) & y_4(0)=\dot{\varphi}_2(0) \end{cases}
$$

Uniqueness of solutions to ordinary differential equations implies that the pendulum motion completely determined by initial values of position and angular speed at t=0.

The Runges-Kutta method to calculate y-values, $\mathbf{y}_{i+1}$ at time $t_{i+1}=t_i+\Delta t$ from values $\mathbf{y}_i$ at time t_i in the system are given by:

$$
\begin{aligned}
k_{11} &= y_{3i} \\
k_{12} &= y_{4i} \\
k_{13} &= f(y_{1i}, y_{2i}, y_{3i}, y_{4i}) \\
k_{14} &= g(y_{1i}, y_{2i}, y_{3i}, y_{4i})
\end{aligned}
$$

$$
\begin{aligned}
k_{21} &= y_{3i} + \tfrac{\Delta t}{2} k_{13} \\
k_{22} &= y_{4i} + \tfrac{\Delta t}{2} k_{14} \\
k_{23} &= f(y_{1i} + \tfrac{\Delta t}{2} k_{11}, y_{2i} + \tfrac{\Delta t}{2} k_{12}, y_{3i} + \tfrac{\Delta t}{2} k_{13}, y_{4i} + \tfrac{\Delta t}{2} k_{14}) \\
k_{24} &= g(y_{1i} + \tfrac{\Delta t}{2} k_{11}, y_{2i} + \tfrac{\Delta t}{2} k_{12}, y_{3i} + \tfrac{\Delta t}{2} k_{13}, y_{4i} + \tfrac{\Delta t}{2} k_{14})
\end{aligned}
$$

$$
\begin{aligned}
k_{31} &= y_{3i} + \tfrac{\Delta t}{2} k_{23} \\
k_{32} &= y_{4i} + \tfrac{\Delta t}{2} k_{24} \\
k_{33} &= f(y_{1i} + \tfrac{\Delta t}{2} k_{21}, y_{2i} + \tfrac{\Delta t}{2} k_{22}, y_{3i} + \tfrac{\Delta t}{2} k_{23}, y_{4i} + \tfrac{\Delta t}{2} k_{24}) \\
k_{34} &= g(y_{1i} + \tfrac{\Delta t}{2} k_{21}, y_{2i} + \tfrac{\Delta t}{2} k_{22}, y_{3i} + \tfrac{\Delta t}{2} k_{23}, y_{4i} + \tfrac{\Delta t}{2} k_{24})
\end{aligned}
$$

$$
\begin{aligned}
k_{41} &= y_{3i} + \Delta t \cdot k_{33} \\
k_{42} &= y_{4i} + \Delta t \cdot k_{34} \\
k_{43} &= f(y_{1i} + \Delta t \cdot k_{31}, y_{2i} + \Delta t \cdot k_{32}, y_{3i} + \Delta t \cdot k_{33}, y_{4i} + \Delta t \cdot k_{34}) \\
k_{44} &= g(y_{1i} + \Delta t \cdot k_{31}, y_{2i} + \Delta t \cdot k_{32}, y_{3i} + \Delta t \cdot k_{33}, y_{4i} + \Delta t \cdot k_{34})
\end{aligned}
$$

$$
\begin{aligned}
y_{1\,i+1} &= y_{1i} + \tfrac{1}{6} \Delta t \cdot (k_{11} + 2k_{21} + 2k_{31} + k_{41}) \\
y_{2\,i+1} &= y_{2i} + \tfrac{1}{6} \Delta t \cdot (k_{12} + 2k_{22} + 2k_{32} + k_{42}) \\
y_{3\,i+1} &= y_{3i} + \tfrac{1}{6} \Delta t \cdot (k_{13} + 2k_{23} + 2k_{33} + k_{43}) \\
y_{4\,i+1} &= y_{4i} + \tfrac{1}{6} \Delta t \cdot (k_{14} + 2k_{24} + 2k_{34} + k_{44})
\end{aligned}
$$