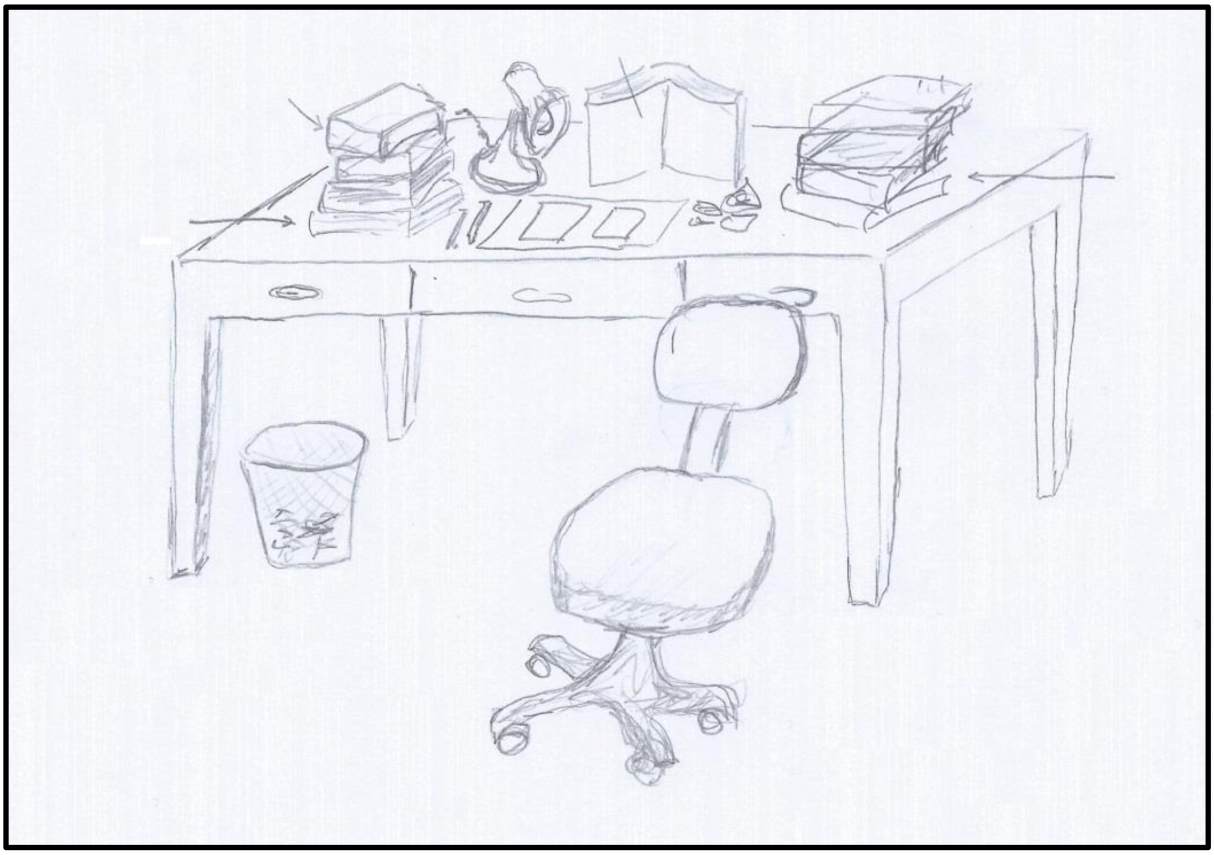
Mathematical Wanderings

Exercises and Solutions



Preface

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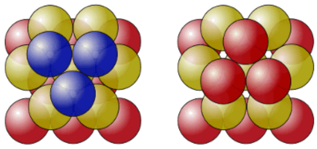
Exercises

Ch.1 Introduction

1.1 Calculate the proportion of space occupied by spheres in the regular packing.



1.2 Write an algorithm or program to derive the period of a fraction

Ch.2 Origins

2.1 Which fractions have a finite sexagesimal expansion?
Calculate the sexagesimal expansion of the first sexagesimally periodic fraction in the series $(1/n)_{n=1}^{\infty}$.

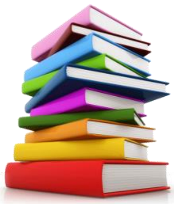
2.2 Pick a number $a_1 > 0$ and let $a_{k+1} = \frac{a_k + N/a_k}{2}$. Show that $\lim_{k\to\infty} a_k = \sqrt{N}$.
This could be the method behind the approximation on YBC 7289.

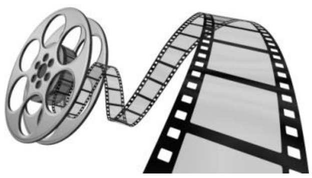
2.3 Show that every fraction can be written as an Egyptian fraction:

$$\frac{p}{q} = N + \sum_{k=1}^{n} \frac{1}{d_k} \quad p, q, N, n, d_k \in \mathbb{Z} \quad \text{and } 1 < d_1 < d_2 < \cdots < d_n$$

2.4 Show that every fraction can be written as an Egyptian fraction in an infinite number of ways.

2.5 Estimate an upper bound for the number of different books, images and movies.



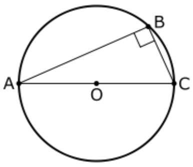


2.6 One person owns seven asava horses, another owns nine haya horses and another owns ten camels. Each gives away two animals, one to each of the others.
They are then equally well off. Find the price of each animal and the total value of the animals possessed by each person. Assume the value of each animal is an integer.
The problem is taken from the Bakshali manuscript.

2.7 Assume you have a method to approximate $\sqrt{x}$, as in Mesopotamia ~1500 BC.
Show a way to approximate $x^{p/q}$ where $x \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}^+ (= \mathbb{N}_1)$.

2.8 In Ramayana, a Sanskrit epic poem one of the characters Ravana sends two spies Shuka and Sarana to estimate the strength of the army of monkeys that builds the land bridge to Sri Lanka. According to Sarana their number is 100 crores of mahaughas A crore is 10^7 and a mahaugha is 10^{60}. How reasonable is Sarana's estimation?

2.9 Prove Thales' theorem:
If A, B and C are points on a circle with diameter AC then angle B is 90º



2.10 Show the inequalities $H \leq G \leq A$ among the Pythagorean means where

$$A = \frac{x+y}{2}\ , G = \sqrt{x \cdot y} \text{ and } H = \left(\frac{1/x+1/y}{2}\right)^{-1} \text{ with } x, y \in \mathbb{R}^+.$$

2.11 Describe the three regular convex n-polytopes of each dimension ≥ 5.

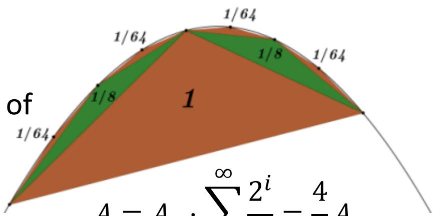
2.12 Express the fraction $\frac{100\,000}{101\,001}$ from chapter one as a continued fraction and show that $[1; 1,1, \dots]$ equals the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

$$\varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}$$

2.13 Derive the area of a disk by using a rectangular decomposition.

2.14 Do what Liu Hui failed to do. Derive the volume of a sphere.

2.15 Show that the area of a parabolic segment can be seen as a sum of areas of inscribed triangles that form a geometric series.



$$A = A_1 \cdot \sum_{1}^{\infty} \frac{2^i}{8^i} = \frac{4}{3} A_1$$

2.16 Solve the cattle problem of Archimedes:

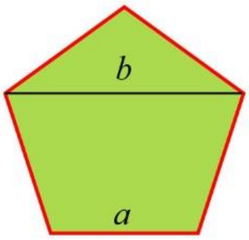
" Compute, O friend the number of cattle of the sun which once grazed upon the plains of Sicily, divided according to color into four herds, …"
They were white, yellow, black and dappled, bulls (W, Y, B, D), cows (w, y, b, d).
There were more bulls than cows and their numbers were as:

$$\begin{aligned} W &= \left(\tfrac{1}{2}+\tfrac{1}{3}\right) B + Y & \qquad w &= \left(\tfrac{1}{3}+\tfrac{1}{4}\right)(B+b) \\ B &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right) D + Y & b &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right)(D+d) \\ D &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right) W + Y & d &= \left(\tfrac{1}{5}+\tfrac{1}{6}\right)(Y+y) \\ & & y &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right)(W+w) \end{aligned}$$

$W + B$ is a square number
$D + Y$ a triangular number

Find the number of bulls and cows of different color, and the total number of cattle.

2.17 Show that the ratio of the diagonal to the side in a regular pentagon equals the golden ratio, $\frac{b}{a} = \varphi \equiv \frac{1+\sqrt{5}}{2}$.



Ch.3 Basics

3.1 Show that a logical n-ary operator $Q(P_1, \dots, P_n)$ with a specified truth table can be given by a formula based on P_i , ¬ and ∧.

	P_1	P_2	$\cdots$	P_{n-1}	P_n	Q	
$(2^n-1)_2$	1	1	$\cdots$	1	1	T_1	
$(2^n-2)_2$	1	1	$\cdots$	1	0	T_2	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$T_i \in \{0,1\}$
$(1)_2$	0	0	$\cdots$	0	1	T_{2^n-1}	
$(0)_2$	0	0	$\cdots$	0	0	T_{2^n}	

3.2 Conway's arrow notation $c_1 \to c_2 \to \cdots \to c_n$ is defined recursively:
1. $p \to q \equiv p^q \quad (p, q \in \mathbb{Z}^+)$
2. $X \to 1 \equiv X \quad$ (X is any chained expression)
3. $X \to p \to (q+1) \equiv \underbrace{X \to (X \to (\cdots (X \to (X) \to q) \cdots) \to q) \to q}_{p \text{ repetitions of } X}$

Knuth's up-arrow notation $a \uparrow^n b$ $(a, b, n \in \mathbb{Z}^+)$ is defined recursively as:
$a \uparrow b = a^b$
$a \uparrow^{n+1} b \equiv \underbrace{a \uparrow^n (a \uparrow^n (\dots \uparrow^n a))}_{b \text{ repetitions of } a}$

Show that:
- Conway chained arrow notation is not an iterated binary operator and
- $p \to q \to r = p \uparrow^r q$
- Express $3 \to 3 \to 3 \to 2$ in Knuth's up-arrow notation.

3.3 Show that a sum of powers of degree p is a polynomial of degree $p+1$ and derive the polynomial $S_p(n)$ for $p = 3$, $p = 4$ and beyond.

$$S_p(n) = \sum_{k=1}^{n} k^p$$

3.4 Prove that if two sets are countable, totally ordered, dense and without upper and lower bounds then they are order-isomorphic.

3.5 Exercises on cardinality of sets:

a) Show that $|\mathbb{R}| = |(0,1)|$.
b) Show that $|\mathcal{P}(A)| > |A|$ for any set A.
c) Show $|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$.
d) Find a bijective function $h: [0,1] \to (0,1)$.

3.6 Prove the binomial identities.

$$\binom{n}{k} = \binom{n}{n-k} \qquad \binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad \binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n \qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

$$\sum_{m=0}^{n}\binom{m}{r} = \binom{n+1}{r+1} \qquad \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$$

3.7 Prove the multinomial theorem:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} {x_1}^{k_1}{x_2}^{k_2} \ldots {x_m}^{k_m}$$

3.8 Stirling numbers $\left\{ n \atop k \right\}$ of the second kind are defined as the number of ways to partition a set of n objects $S_n = \{1,2, \ldots, n\}$ into k non-empty subsets. Show that k!$\left\{ n \atop k \right\}$ equals the number of surjective functions $f\colon S_n \to S_k$ and that

$$\left\{ n \atop k \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$$

3.9

According to legend there is a temple with monks and 64 golden disks resting on three pillars. Ancient rules dictate that a disk may never rest on a smaller disk. When all disks have been moved the world will end. They are working day and night moving one disk every second. What is the shortest time to move all 64 golden disks?

3.10 How many different messages of length n can be built from two symbols of length 1 and length 2?

$n = 1$, {  }

$n = 2$, {  ,  }

Message of length 12

Compare the growth rate with a geometric sequence.

3.11 Prove Euler-Hierholzer's theorem from graph theory. A connected graph has an Euler cycle if and only if every node is of even degree.

3.12 Show that the set of numbers $\mathbb{Q}[\sqrt{2}] \coloneqq \{q_1 + q_2\sqrt{2} | q_1, q_2 \in \mathbb{Q}\}$ form a field under ordinary addition and multiplication.

3.13 Show equivalence of the different definitions of multiplicity k for roots of $P(z)$.

$$\begin{matrix} (z-\alpha)^k | P(z) \\ (z-\alpha)^{k+1} \nmid P(z) \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} P^{(i)}(\alpha) = 0 \ \text{ for } i \in \{0,1,\dots,k-1\} \\ P^{(k)}(\alpha) \neq 0 \end{matrix}$$

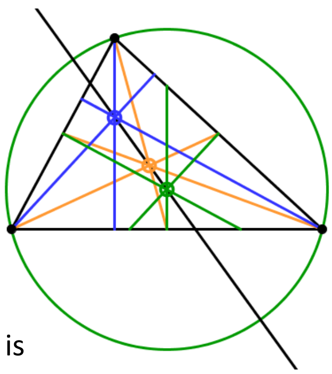
3.14 The location of a pirate treasure is described as follows:
Go from the gallows to the oak, turn 90 degrees to the left, walk the same distance and put a knife in the ground. Go back to the gallows, walk to the pine, turn 90 degrees to the right, walk the same distance and put another knife in the ground. Midway between the knives, dig and you will find the treasure.

Descendants of the pirate found the description. They went to the island and found the pine and the oak but no gallows but still they could find the treasure. Describe where they found it.

3.15 Show $e^z e^w = e^{z+w}$ for $z, w \in \mathbb{C}$.

3.16 A Graeco-Latin square or an Euler square of order n is an arrangement of symbols from $\mathrm{G} = \{\alpha, \beta, \gamma, \dots\}$ and $\mathrm{L} = \{a, b, c, \dots\}$ with $|\mathrm{G}|=|\mathrm{L}|=n$ in such a way that each cell of an $n \times n$ square contains an ordered pair $(g, l) \in \mathrm{G} \times \mathrm{L}$. Every row and every column contain each element of G and each element of L exactly once and no cells contain the same pair. Euler presented the problem for $n = 6$ with $\mathrm{G} = \{\text{officer ranks}\}$ and $\mathrm{L} = \{\text{regiments}\}$, "the thirty-six officers' problem". He constructed Graeco-Latin squares for n=$2k+1$ and n=$4k$ Euler conjectured that no Graeco-Latin squares exists for n=$4k+2$. Show that he was wrong!

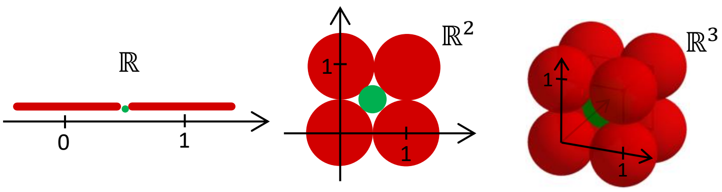
A similar problem with n=4 and 16 playing cards, $\mathrm{G} = \{\mathrm{A}, \mathrm{K}, \mathrm{Q}, \mathrm{J}\}$ and $\mathrm{L} = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ has an extra constraint. Each diagonal should also contain all four face values and all four suits. How many solutions are there?



3.17 Show that the three altitudes of a triangle have one point in common, (the orthocenter).

3.18 Show that orthocenter, centroid and circumcenter of a non-equilateral triangle are collinear.

3.19. Explore how the radius varies with dimension of a sphere that is squeezed in between spheres centered at integer coordinates $\mathbb{Z}^n$ in $\mathbb{R}^n$.



3.20 $f: X \to Y$ is a function between two metric spaces with $\|a-b\| = d(a,b)$
Show that the following definitions of $\lim\limits_{x \to x_0} f(x) = y_0$ are equivalent.
A. For every $\epsilon > 0$ there is a $\delta > 0$ such that: $0 < \|x - x_0\|_X < \delta \Longrightarrow \|f(x) - y_0\|_Y < \epsilon$
B. For every neighborhood $\mathcal{V}$ of $y_0$there is a punctured neighborhood $\mathcal{U}$ of x_0 s.t. $f(\mathcal{U}) \subseteq \mathcal{V}$

3.21 Show that if $\lim\limits_{x \to c} f(x) = A$ and $\lim\limits_{x \to c} g(x) = B$ then

$a)$ $\lim\limits_{x \to c}(f(x) \cdot g(x)) = A \cdot B$
$b)$ $\lim\limits_{x \to c}(f(x)/g(x)) = A/B$ if $B \neq 0$

3.22 ?

3.23 Is there a function $f \in \mathrm{C}^0(\mathbb{R})$ such that f is continuous on $\mathbb{Q}$ but not on $\mathbb{R} \setminus \mathbb{Q}$?

3.24 Prove the Archimedean property for $\mathbb{R}$:
There is no positive real pair x, y such that $n \cdot x < y$ for every $n \in \mathbb{N}$.

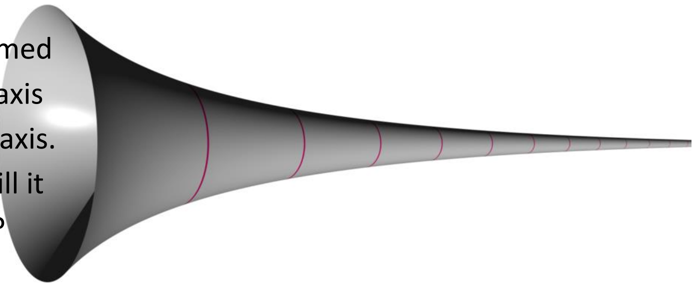
3.25 Prove that if f is continuous on a compact interval $[a, b]$ then
f is uniformly continuous on that interval.

3.26. Assume that $f: [a, b] \to [c, d]$ is continuous and invertible and that f^{-1} is differentiable.
Show that: $\int f^{-1}(y)dy = y \cdot f^{-1}(y) - F \circ f^{-1}(y) + C$
Give the equation a figurative interpretation, a proof without words.

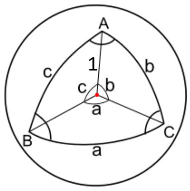
3.27 Show that the Cantor function also known as the Devil's staircase c: $[0,1] \to [0,1]$
is increasing, surjective, continuous and has a graph of arc length 2.
$c(x)$ is defined by:
Express x in base 3 and replace all digits after the first digit=1 (if any) with zeros and
replace all digits=2 after this with digits=1 and reinterpret the sequence as base 2 to get $c(x)$.

3.28 Show that the area $\int_\alpha^\beta \mathrm{f(x)\,dx}$ for $\mathrm{f(x)} = 1/\mathrm{x}$ is unaffected
by a rescaling of boundaries $[\alpha, \beta] \curvearrowright [c\alpha, c\beta]$. $\alpha, \beta, c \in \mathbb{R}^+$

3.29 Calculate the volume and area formed
by rotating $\mathrm{y} = 1/\mathrm{x}$ around the x-axis
for the interval $[1, \infty)$ along the x-axis.
How much paint would it take to fill it
and how much to paint the inside?

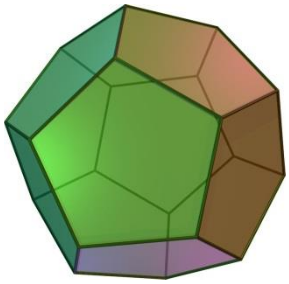


3.30 Show that the spherical law of cosines
$\cos c = \cos a \cos b + \sin a \sin b \cos C$
reduces to the planar law of cosines
$c^2 = a^2 + b^2 - 2ab \cos C$ as $a, b, c \to 0$.



3.31 A pyramid has an equilateral triangle as base, the sides are isosceles triangles and the height of the pyramid equals the distance between the height and the base. What is the angle between two sides?

3.32 Prove the spherical law of cosines
$\cos c = \cos a \cos b + \sin a \sin b \cos C$.



3.33 Calculate the inner angle between adjacent faces in a regular dodecahedron with regular pentagons as faces. (dodecahedron from Greek, do-2 deca-10 →12 faces)

3.34 Use the definition of the hyperbolic functions from a hyperbola to show
$\cosh A = \frac{1}{2}(e^A + e^{-A})$ $\quad$ $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$
$\sinh A = \frac{1}{2}(e^A - e^{-A})$ $\quad$ $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

3.35 Derive the Taylor series expansions of $\ln(\mathrm{x} + 1)$, $\arctan \mathrm{x}$ and $\operatorname{artanh} \mathrm{x}$ around $\mathrm{x} = 0$ and show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

3.36 Calculate $f_\omega(3)$ and show that $\mathrm{f}_{\omega^2}(\mathrm{n}) > \mathrm{n} \to \cdots \to \mathrm{n}$ (n n's)
f_α comes from the fast-growing hierarchy.
$f_0(n) = n + 1$
$f_{\alpha+1}(n) = f_\alpha^n(n)$
$f_\alpha(n) = f_{\alpha_n}(n)$ when $\alpha = \lim_n \alpha_n$ is a limit ordinal.

3.37 A real or complex series $\sum_{\mathrm{k}=0}^{\infty} \mathrm{a_k}$ is said to be absolutely convergent if $\mathrm{S_n} = \sum_{\mathrm{k}=0}^{\mathrm{n}} |\mathrm{a_k}|$ is limited ($\sum_{\mathrm{k}=0}^{\infty} |\mathrm{a_k}| = \sup\{\mathrm{S_n} | \mathrm{n} \in \mathbb{N}_0\} = \mathrm{S}$).
A series $\sum_{\mathrm{k}=0}^{\infty} \mathrm{b_k}$ that is convergent $\left(\lim_{\mathrm{n}\to\infty} (\sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{b_k}) \in \mathbb{C}\right)$ without being absolutely convergent is conditionally convergent. Show that:

I. Absolute convergence $\Longrightarrow$ convergence.
II. Sum of absolutely convergent series is independent of the ordering order of the terms.
III. The sum of a real conditionally convergent series can attain any real number with an appropriate summation order.

3.38. Show that every solution to $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_0 y = 0$
with characteristic polynomial $l(r) = \prod_{k=1}^{\nu}(r - r_k)^{n_k}$ is of the form
$y(x) = \sum_{k=1}^{\nu} P_k(x)e^{r_k x}$ with $\deg P_k < n_k$.

3.39 Homogeneous linear recurrence relation with constant coefficients of order n:

$$a_k = c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_n a_{k-n} \ (*) \quad a_i, c_i, r_i \in \mathbb{C}$$

with characteristic polynomial $p(t) = t^n - \sum_{i=1}^{n} c_i t^{n-i} = \prod_{j=1}^{\nu}(t - r_j)^{n_j}$, $n_1 + \cdots + n_\nu = n$
Show that $a_k = \sum_{j=1}^{\nu} P_j(k){r_j}^k$ with polynomials P_j of degree less than n_j solves $(*)$.

3.40 The weighted power mean $M_p$of $x_1, \dots, x_n \in \mathbb{R}^+$
with weights $w_i \in \mathbb{R}^+$ and $\sum_{i=1}^{n} w_i = 1$ is defined by

$M_p(x_1, \dots, x_n) = (\sum_{i=1}^{n} w_i {x_i}^p)^{1/p}$ for $p \in \mathbb{R} \setminus \{0\}$
$M_0(x_1, \dots, x_n) = \prod_{i=1}^{n} {x_i}^{w_i}$
$M_{-\infty}(x_1, \dots, x_n) = \min(x_1, \dots, x_n)$
$M_{\infty}(x_1, \dots, x_n) = \max(x_1, \dots, x_n)$

$p = -1$: harmonic mean
$p = 0$: geometric mean
$p = 1$: arithmetic mean
$p = 2$: square mean

Show:
$\lim_{p \to 0} M_p = M_0$
$\lim_{p \to -\infty} M_p = M_{-\infty}$
$\lim_{p \to \infty} M_p = M_{\infty}$
$p < q \implies M_p(x_1, \dots, x_n) \le M_q(x_1, \dots, x_n)$ with equality iff $x_1 = x_2 = \cdots = x_n$
($\min \le H.M \le G.M \le A.M \le S.M \le \max$)

Ch.4 Return

4.1 Prove $p^2 | (2^{p(p-1)} - 1)$ with p a prime.

4.2 Show that if a fraction a/p with $0 < a < p$ and p a prime has a decimal expansion with even period $a/p = 0.\overline{r_1 \dots r_n r_{n+1} \dots r_{2n}}$ then $r_i + r_{i+n} = 9$

Example: $\frac{1}{17} = 0.\overline{\underbrace{05882352}_{A}\,\underbrace{94117647}_{B}}$ $\quad \begin{array}{r} 05882352 \\ \underline{94117647} \\ 99999999 \end{array} \quad A + B = 10^n - 1$

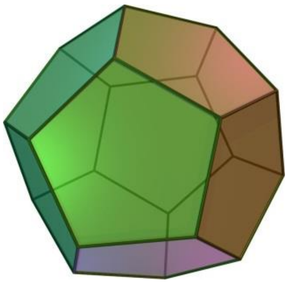
4.3. There are infinitely many triples of positive integers (a, b, c) with $\gcd(a, b, c) = 1$ s.t. $a + b = c$ and $q(a, b, c) > 1$ where:

$$q(a, b, c) = \frac{\log(c)}{\log\big(\mathrm{rad}(abc)\big)} \qquad , \qquad \mathrm{rad}\left(\prod p_i^{k_i}\right) = \prod p_i$$

The abc-conjecture states: $\varepsilon > 0 \Rightarrow$ only finitely many triples has $q(a, b, c) > 1 + \varepsilon$.
If the abc-conjecture is true then there is a maximal value of $q(a, b, c)$.
Assume the abc-conjecture and that $q(a, b, c)$ is always less than 2.
Show that Fermat's last theorem $a^n + b^n = c^n$ with $\gcd(a, b, c) = 1$ holds for $n \geq 6$.
(The abc-conjecture says nothing about the limit of $q(a, b, c)$, biggest known case is 1.63.)

Ch.5 History

Ch.6 Linear algebra

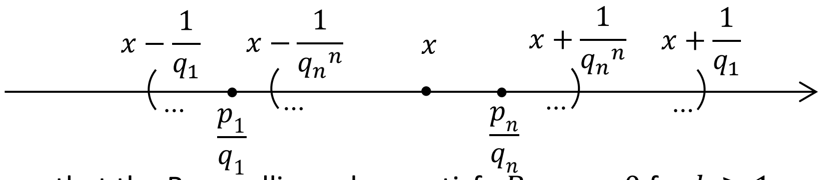


6.x Show that the inner angle between two adjacent faces in a regular dodecahedron equals $2\arctan(\varphi)$ where $\varphi \equiv (\sqrt{5}+1)/2$ is the golden ratio.
Solve it by using a matrix for rotation.

Appendix C

C.1 Show that the Liouville numbers $\mathbb{L}$ are transcendental and that they form an uncountable dense subset of $\mathbb{R}$ with Lebesgue measure zero.

$$\mathbb{L} = \left\{x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N}_1 \, \exists (p,q) \in \mathbb{Z} \times \mathbb{N}_2 \left(\left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right) \right\} \qquad \mathbb{N}_k = \{k, k+1, \dots\}$$



C.2 Show that the Bernoulli numbers satisfy $B_{2k+1} = 0$ for $k \geq 1$.

C.3 Prove that ordinary and binomial convolutions,

$$\langle f_n \rangle \star \langle g_n \rangle = \langle \sum_{k=0}^{n} f_k g_{n-k} \rangle \text{ and } \langle f_n \rangle \star^b \langle g_n \rangle = \langle \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k} \rangle$$

are commutative and associative operators with identity $\langle 1,0,0,\dots \rangle$ and have a unique inverse for sequences $\langle a_0, a_1, a_2, \dots \rangle$ with $a_0 \neq 0$.

C.4 Use the formula for the resultant, $\Delta(P) = (-1)^{n(n-1)/2} R(P,P')/a_n$ of $P{=}a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = a_n (z - r_1)(z - r_2) \dots (z - r_n)$ with $R(P,Q) = |S_{P,Q}|$ where $S_{P,Q}$ is the Sylvester matrix to find the discriminant of $ax^4 + bx^3 + cx^2 + dx + e$ and check that the result is in accordance with the definition $\Delta(P) \equiv a_n^{2n-2} \cdot \prod_{1 \leq i < j \leq n} \left(r_i - r_j\right)^2$.

C.5. The resultant $R(f,g)$ of two polynomials with coefficients in a field $\mathbb{F}$ where $f(x) = a_n x^n + \dots + a_0$ and $g(x) = b_m x^m + \dots + b_0$, with roots $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ in the algebraic closure of $\mathbb{F}$ can be defined in two alternate ways:

1. $R(f,g) \equiv a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} \left(\alpha_i - \beta_j\right)$

2. $$R(f,g) \equiv \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & b_2 & b_1 & b_0 \end{vmatrix}$$

Show that the two definitions are equivalent.

Hints

Ch.1 Introduction

1.1 Calculate the distance between successive layers and make a regular tiling of each layer. The answer is $\pi/(3\sqrt{2})$

1.2 Extend the algorithm given for long division.

Ch.2 Origins

2.1 Read about representation and base in chapter 3.

2.2 Look at how the series a_k is generated in a coordinate system with two graphs drawn: $y = (x + N/x)/2$ and $y = x$,
or study the Newton-Raphson method for finding roots.

2.3 Use the greedy algorithm.

2.4 Derive a new Egyptian fraction from an existing one.

2.5 Assume a book is a string of symbols from a list of 100 symbols on no more than 1000 pages. An image is roughly 1000×1000 pixel, each pixel described by a red, green and blue intensity between 0 and FF_{16}.
Assume a movie is less than 100 fps and lasts no more than three hours.

2.6 Express the unknown prices in terms of their equal wealth.

2.7 Use a binary decimal expansion.

2.8. Estimate the total volume, mass or area of the monkey army compared to the earth op to the land bridge to Sri Lanka.

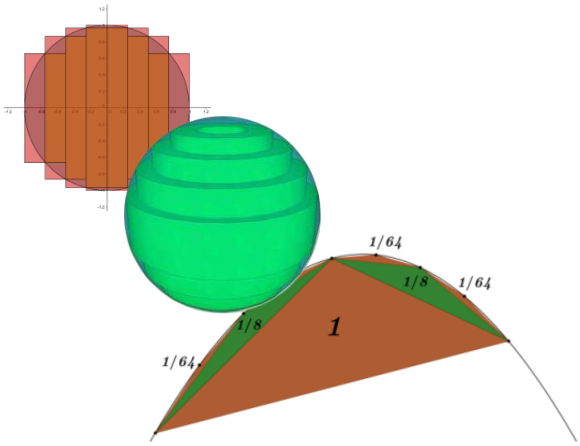
2.9 Draw an extra radius.

2.10 Use the geometrical constructions of Pythagorean means on page 47 in *MW*.

2.11 Look at the figures on the bottom of page 52 in *MW*.

2.12 Look at page 60 in *MW*.

2.13 Use integration.

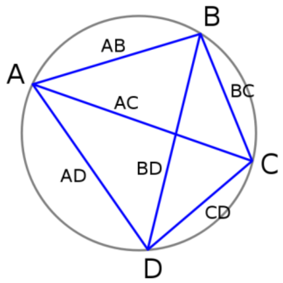


2.14 Use cylindrical shells.

2.15 Old points are the mean to get new points.

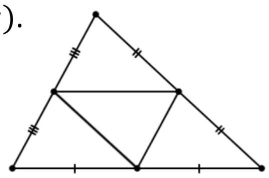
2.16 Use a program like Mathematica or MATLAB

2.17 Prove and use Ptolemy's theorem for a cyclic quadrilateral:
$|\overline{\mathrm{AC}}| \cdot |\overline{BD}| = |\overline{\mathrm{AB}}| \cdot |\overline{CD}| + |\overline{\mathrm{AD}}| \cdot |\overline{BC}|$



Ch.3 Basics

3.2 Show that $a \to b \to c$ is neither left-associative $(a \to b) \to c$ nor right-associative $a \to (b \to c)$.



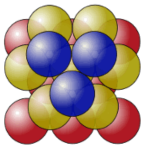
3.18 Draw a midpoint triangle.

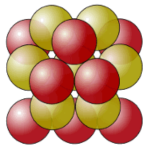
3.19 Use the space diagonal.

3.x Look at the cross-section of the dodecahedron.

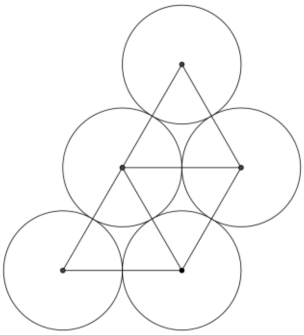
Solutions

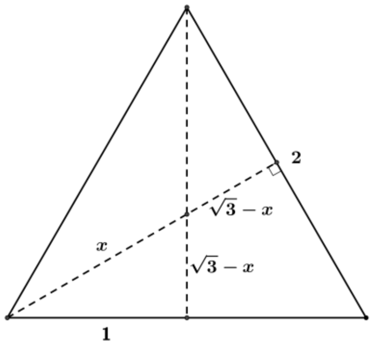
1.1 Calculate the proportion of space occupied by spheres in the regular packing.

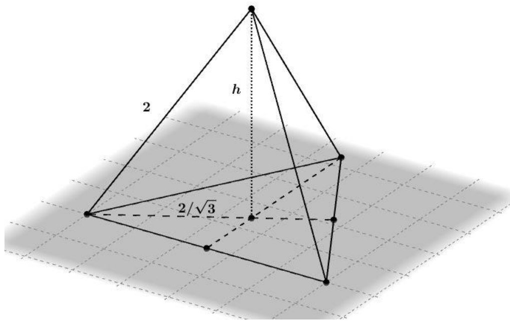




The problem is scale invariant so we can assume spheres of radius one.
Tile the plane with equilateral triangles.
The distance between planes equals the height in a tetrahedron with side two.







$$\begin{aligned} 1^2 + \left(\sqrt{3}-x\right)^2 &= x^2 \\ x &= 2/\sqrt{3} \end{aligned}$$

$$\begin{aligned} 4/3 + h^2 &= 4 \\ h &= 2\sqrt{2}/\sqrt{3} \end{aligned}$$

Space is tiled by prisms of base area $\sqrt{3}$ and height $2\sqrt{2}/\sqrt{3}$, the volume is $2\sqrt{2}$.
Each prism contains spheres of volume $3 \cdot \frac{1}{6} \cdot \frac{4\pi 1^3}{3} = 2\pi/3$
Proportion occupied by spheres: $\dfrac{\pi}{3\sqrt{2}}$

1.2 Write an algorithm or program to derive the period of a fraction.

Algorithm

```
Read data, P and Q   (P/Q)
0→J
Initiate N, R and lists NList, RList
If  R=0
  Print "Q divides P"
Else
  False→Repeat
  While R≠0 and not Repet
  J+1→J
  Initiate N, R and lists NList, RList
  If  R≠0  // Test if R is previous in list
    1→I
    False → Repeat
    While I<J and L_R(I)≠R
      I+1→I
    If I<J
      True → Repeat
If R=0
  Print "No repetition P/Q=" , P/Q
Else
  Print "Integer part= ", L_N(1)
  Print "Initial decimals: " , L_N( 2..I )
  Print "Repeat digits: ", L_N( I+1..J)
  Print "Period equals " , J-I
```

Algorithm for calculator

TI-83 Program:Period

```
ClrList L_N,L_R
Clrhome
Input “NUMERATOR(P)”, P
Input “DENOMINATOR(Q)”, Q
1→J
int(P/Q)→N
P-N*Q→R
N→L_N(J)
R→L_R(J)
If R=0
Then
Disp “Q DIVIDES P”
Else
0→A (A Boolean: is R contained in R_List)
While R≠0 and not(A)
J+1→J
R*10→R
int(R/Q)→N
R-N*Q→R
N→L_N(J)
R→L_R(J)
If R≠0
Then
1→I
0→A
While I<J and L_R(I)≠R
I+1→I
End
If I<J: 1→A
End
End
If R=0
Then
Disp “NO REPEAT P/Q= “,P/Q
Else
Disp “INTEGER PART= “, L_N(1)
If I>1
Then
Disp “INITIAL DECIMALS: ”, seq(L_N(X),X,2,I)
End
Disp “PERIOD IS “,J-I
Disp “REPEAT DIGITS: ”,seq(L_N(X),X,I+1,J)
End
End
```

2.1 Which fractions have a finite sexagesimal expansion?
Calculate the sexagesimal expansion of the first sexagesimally periodic fraction in the series $(1/n)_{n=1}^{\infty}$.

Assume a positive fraction $\frac{p}{q}$ with p and q without common divisor $(p, q) = 1$.

$$\frac{p}{q} \text{ has finite expansion in base B } \Longrightarrow$$
$$\frac{p}{q} = N + \sum_{k=1}^{n} a_k \cdot \frac{1}{B^k} \quad \text{with n} < \infty \quad \Longrightarrow$$
$$\frac{p}{q} \cdot B^n = M \quad \text{for some} \quad \text{n}, \text{M} \in \mathbb{Z}$$

If q has some prime factor not in B this is not possible ⇒ infinite expansion.
If q has no prime factor not in B, we can write $\frac{p}{q} \cdot B^n = M$ for some $n, M \in \mathbb{Z}$ ⇒ finite expansion.
For sexagesimal expansion $B = 2^2 \cdot 3 \cdot 5$.

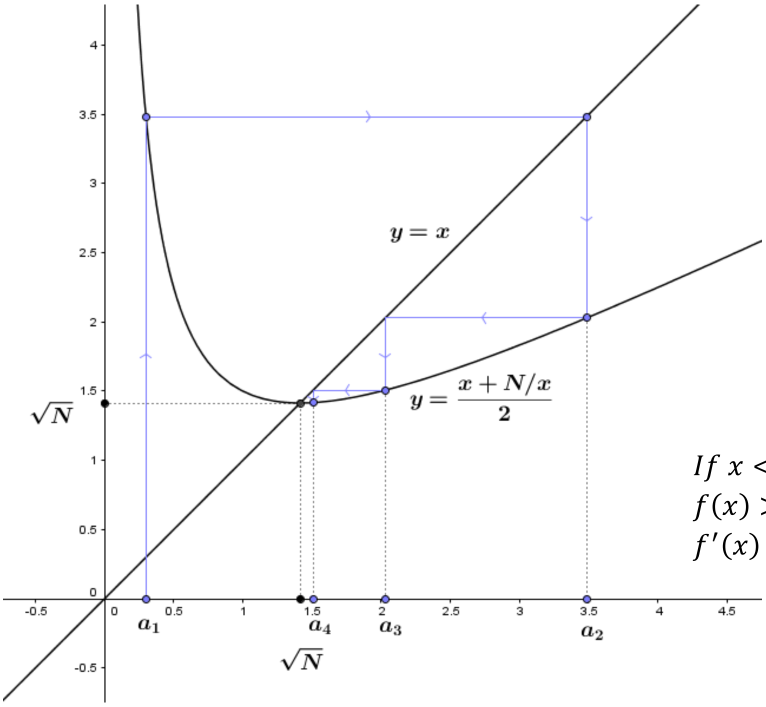
∴ Fractions p/q with $(p, q) = 1$ have finite sexagesimal expansion if $q = 2^a 3^b 5^c$ with $a, b, c \in \mathbb{N}_0$

The first fraction with an infinite expansion in the series $(1/n)_{n=1}^{\infty}$ will be 1/7.
The long division in chapter one shows that for all bases:
If p/q has infinite expansion it will be periodic with period less than q.

	0.	8	34	17	
7	1.	0	0	0	
	0				
	1	0			(=60)
		56			
		4	0		(=240)
		3	58		
			2	0	(=120)
			1	59	
				1	

$$\frac{1}{7} = 0.\overline{8\ \ 34\ \ 17}$$

2.2 Pick a number $a_1 > 0$ and let $a_{k+1} = \frac{a_k + N/a_k}{2}$. Show that $\lim_{k\to\infty} a_k = \sqrt{N}$.
This could be the method behind the approximation on YBC 7289.



$f(x) = \frac{1}{2}(x + N/x)$
$f'(x) = \frac{1}{2}(1 - N/x^2)$

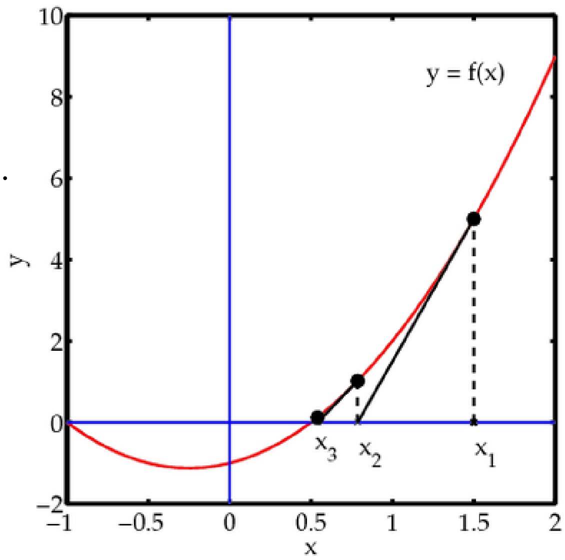
$If\ x < \sqrt{N}$	$If\ x = \sqrt{N}$	$If\ x > \sqrt{N}$
$f(x) > x$	$f(x) = x$	$f(x) < x$
$f'(x) < 0$	$f'(x) = 0$	$f'(x) > 0$

If $a_1 = \sqrt{N}$ then $a_k = \sqrt{N}$ for all k so $\lim_{k\to\infty} a_k = \sqrt{N}$. Assume $a_k \neq \sqrt{N}$.
It is clear that $(a_k)_{k=2}^{\infty}$ is a strictly decreasing sequence bounded from below by $\sqrt{N}$.
The limit must exist so $\lim_{k\to\infty} a_k = a$ with $a \geq \sqrt{N}$.

$\xleftarrow{\delta_\varepsilon}$

$\sqrt{N} \quad a \quad f(a+\varepsilon) \quad a+\varepsilon$

Assume $> \sqrt{N}$
$\lim_{\varepsilon\to 0} \underbrace{[(a+\varepsilon) - f(a+\varepsilon)]}_{\delta_\varepsilon} = \frac{a}{2} - \frac{N}{2a} = \delta > 0.$
If we make k large enough we get a contradiction $a_k < a$ so $a \ngtr \sqrt{N}$.
$\therefore \lim_{k\to\infty} a_k = \sqrt{N}$



Alternative solution:
Let $f(x) = x^2 - N$, with root $\sqrt{N}$.
Apply Newton-Raphson method to find the root of $f(x)$.

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{{x_n}^2 - N}{2x_n} \\ &= \frac{x_n + N/x_n}{2} \end{aligned}$$

It is not known how the Babylonians found their approximation to $\sqrt{2}$ on the clay tablet YBC 7289.
Maybe they used $a_1 = 1.5$ and $a_{k+1} = \frac{a_k + 2/a_k}{2}$, three iterations would be enough.

2.3 Show that every fraction can be written as an Egyptian fraction:
$\frac{p}{q} = N + \sum_{k=1}^{n} \frac{1}{d_k}$ $p, q, N, n, d_k \in \mathbb{Z}$ and $1 < d_1 < d_2 < \cdots < d_n$

Let us assume that $p, q > 0$ and $p < q$ and concentrate on showing:

$$\frac{p}{q} = \sum_{k=1}^{n} \frac{1}{d_k} \quad \text{with} \quad 1 < d_1 < d_2 < \cdots < d_n$$

If $p = 1$ we are done.
If $p > 1$, use the greedy algorithm which means,
pick the largest possible unit fraction, and keep doing that to what remains.

$\frac{p}{q} = \frac{1}{\lceil q/p \rceil} + \frac{(-q) \bmod p}{q \lceil q/p \rceil}$

Ex: $-5 \bmod 3 \in \{-5, -5+3, -5+2\cdot 3, \dots\}$
$(-q) \bmod p = -q + n \cdot p$ where
$n \in \mathbb{N}_1 = \{1,2,\dots\}$ and $-q + n \cdot p \in \{0,1,\dots,(p-1)\}$
$\Rightarrow n = \lceil q/p \rceil$

Right-hand side (RHS) equals:

$\frac{q + (-q) \bmod p}{q \lceil q/p \rceil} = \frac{q + (-q) + \lceil q/p \rceil p}{q \lceil q/p \rceil} = \frac{p}{q} = LHS$

$0 \le (-q) \bmod p < \mathrm{p}$
If $(-q) \bmod p = 0$, then we are done.
If not we can reapply the expansion with a strictly lower numerator so
the process will end in a finite number of steps.

Show $d_{n+1} > d_n$:
Assume not, then $\frac{1}{d_n} + \frac{1}{d_{n+1}} \ge \frac{2}{d_n}$ but $\frac{1}{d_n - 1} \le \frac{2}{d_n}$ if $d_n \ge 2$. So we could have had $d_n - 1$ instead of d_n.
d_n must be strictly increasing.

For a general fraction $\frac{p}{q}$ we get:

$$\frac{p}{q} = N + \sum_{k=1}^{n} \frac{1}{d_k} \quad \text{with} \quad 1 < d_1 < d_2 < \cdots < d_n$$

2.4 Show that every fraction $\frac{p}{q}$ $(0 < p < q)$ can be written as an Egyptian fraction in an infinite number of ways.

Exercise 2.3 shows that there is at least one decomposition into Egyptian fraction
$\frac{p}{q} = \sum_{k=1}^{n} 1/d_k \quad (1 < d_1 < d_2 < \cdots < d_n)$
If the number is finite we can put them in a list and choose a decomposition with a maximal denominator D_n.

$$\frac{p}{q} = \sum_{k=1}^{n} \frac{1}{D_k} \quad (1 < D_1 < D_2 < \cdots < D_n)$$

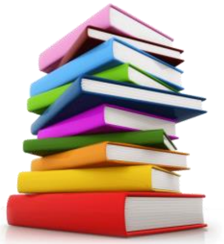
$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} \Longrightarrow \frac{p}{q} \quad = \sum_{k=1}^{n-1} \frac{1}{D_k} + \frac{1/2 + 1/3 + 1/6}{D_n}$$

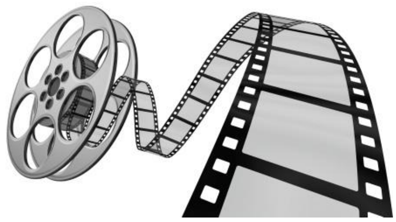
$$= \sum_{k=1}^{n-1} \frac{1}{D_k} + \frac{1}{2D_n} + \frac{1}{3D_n} + \frac{1}{6D_n}$$

This is a new decomposition not in the list since $6D_n > D_n$ so the list can't be finite.

∴ Every fraction can be divided into Egyptian fractions in an infinite number of ways.

2.5 Estimate an upper bound for the number of different books, images and movies.





Let us assume that a book is identified by a string of symbols taken from a list of 100 symbols written on 1000 pages with 40 rows and 70 symbols per row.
All books of shorter length are effectively included since a blank is part of our list of symbols.

$\boxed{\text{Total number of combinations for books: } 100^{1000\cdot 40\cdot 70} = (10^2)^{280\,000} = 10^{560\,000} = 10^{5.6\cdot 10^5}}$

The number is quite a bit larger than a googol 10^{100}, but much smaller than a googolplex $10^{10^{100}}$.

It exceeds the biggest number in the old Greek nomenclature for numbers, a myriad myriads 10^8, but it is far smaller than the biggest number named by Archimedes, $10^{8\cdot 10^{16}}$ in *The Sand Reckoner*.

It is puny compared to "the incalculable" $10^{7\cdot 2^{122}} \approx 10^{3.7\cdot 10^{37}}$ from the Mahayana Buddhist scripture *Buddha-avatamsaka-nama-vaipulya-sutra* (Flower Garland Sutra of Great Universal Buddha)
(Japanese: ふかせつふかせつてん hukasetsuhukasetsuten , Chinese: 不可說不可說轉 Bukeshuo bukeshuo zhuan)

Let us assume that an image has a size of 1000×1000 pixels and that the color of each pixel is described by three numbers for red, green and blue intensity ranging from 0 to FF_{16}=255 which gives $2^{8\cdot 3}$ different color values for each pixel.

$\boxed{\text{Total number of combinations for images: } (2^{24})^{10^6} = 10^{\log 2\cdot 24\cdot 10^6} \approx 10^{7.2\cdot 10^6}}$

In round figures the same number as for books.

Let us assume that a movie is a sequence of images 1000×1000 pixels, 100 frames per second (fps), going on for no longer than three hours and that there is no sound.
(Standard film format 24fps, TV up to 100fps, 300fps have been tested for sport to enable high quality slow motion capture)

$\boxed{\text{Total number of combinations for movies: } \left((2^{24})^{(10^6)}\right)^{3\cdot 3600\cdot 100} \approx 10^{7.8\cdot 10^{12}}}$

Still much smaller than the googolplex, the maximum in *The Sand Reckoner* and "the incalculable".

2.6 One person owns seven asava horses, another owns nine haya horses and another owns ten camels. Each gives away two animals, one to each of the others. They are then equally well off. Find the price of each animal and the total value of the animals possessed by each person. Assume the value of each animal is an integer, in a suitable currency called coin. (The problem is taken from the Bakshali manuscript.)

Assume:

$$\begin{array}{rl} \text{Asava horse:} & x \text{ coins} \\ \text{Haya horse:} & y \text{ coins} \\ \text{Camel:} & z \text{ coins} \end{array} \rightarrow 5x + y + z = x + 7y + z = x + y + 8z = w$$

Express x, y and z in terms of the common wealth w.

$$\begin{array}{l} x = aw \\ y = bw \\ z = cw \end{array} \rightarrow \begin{cases} 5a + b + c = 1 \\ a + 7b + c = 1 \\ a + b + 8c = 1 \end{cases} \Leftrightarrow \begin{cases} a = 21/131 \\ b = 14/131 \\ c = 12/131 \end{cases} \Leftrightarrow \begin{cases} x = 21w/131 \\ y = 14w/131 \\ z = 12w/131 \end{cases}$$

$$x, y, z \in \mathbb{Z}^+ \implies w = 131n \quad , \ n \in \{1,2,3 \dots\}$$

$$\boxed{\begin{array}{rl} \text{Asava horse:} & 21n \text{ coins} \\ \text{Haya horse:} & 14n \text{ coins} \\ \text{Camel:} & 12n \text{ coins} \\ \text{After donations, they each own:} & 131n \text{ coins} \end{array}}$$

(The solution given in the Bakhshali manuscript corresponds to $n = 2$.)

2.7 Assume you have a method to approximate $\sqrt{x}$, as in Mesopotamia ~1500 BC
Show a way to approximate $x^{p/q}$ where $x \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}^+ (= \mathbb{N}_1)$

$$x^{p/q} = x^{N + \sum_{n=1}^{\infty} d_n \cdot 2^{-n}} = x^N \cdot \sqrt{x}^{d_1} \cdot \sqrt{\sqrt{x}}^{d_2} \cdot \sqrt{\sqrt{\sqrt{x}}}^{d_3} \cdot \dots = x^N \cdot \prod_{\{n | d_n = 1\}} f^n(x) \quad \text{with } f(x) = \sqrt{x}$$

$$\boxed{\begin{array}{l} N = \lfloor p/q \rfloor \in \mathbb{N}_0 \\ d_n \in \{0,1\} \end{array}} \qquad \boxed{\begin{array}{l} f^1(x) = \sqrt{x} \\ f^n(x) = f(f^{n-1}(x)) \end{array}}$$

Having an approximation for $\sqrt{x}$ we can approximate $\sqrt{\sqrt{x}}$ and so on.
From a numerical point of view the method is not very good due to slow convergence and accumulating errors in repeated square root approximations to calculate $x^{1/2^n}$.

2.8 In Ramayana, a Sanskrit epic poem one of the characters Ravana sends two spies Shuka and Sarana to estimate the strength of the army of monkeys that builds the land bridge to Sri Lanka. According to Sarana their number is 100 crores of mahaughas A crore is 10^7 and a mahaugha is 10^{60}. How reasonable is Sarana's estimation?

Assume a monkey weighs 50 kg and has a volume of $50\ \text{kg}/(1\ \text{kg} \cdot \text{dm}^{-3}) = 0.05\ \text{m}^3$

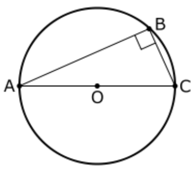
Mass of 10^{69} monkeys:	$5.0 \cdot 10^{70}$ kg	Volume of 10^{69} monkeys:	$5.0 \cdot 10^{67}\text{m}^3$
Mass of earth:	$6.0 \cdot 10^{24}$ kg	Volume of earth:	$1.1 \cdot 10^{21}\ \text{m}^3$
Mass of observable universe:	$3.4 \cdot 10^{54}$ kg	Volume of observable universe:	$3.4 \cdot 10^{80}\ \text{m}^3$

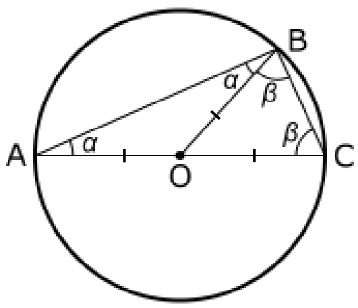
Assume that the Sri Lanka land bridge is 100 km long and 100 meter wide.
Putting all the monkeys on the bridge, and standing on top of each other the monkeys would form a wall higher than $5.0 \cdot 10^{67}\text{m}^3/\ (100\ \text{km} \cdot 100\text{m}) = 5.0 \cdot 10^{60}$ m.

The distance to the moon is $4 \cdot 10^8$ m and the diameter of observable universe is $9 \cdot 10^{26}$ m.

It seems that enormous numbers in Hindu and Buddhist texts were not used for calculations but more as a way of expressing that some quantity or timespan was really big.

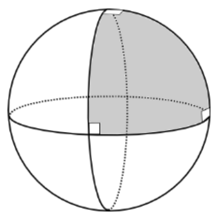
2.9 Prove Thales' theorem:
If A, B and C are points on a circle with diameter AC then angle B is 90º

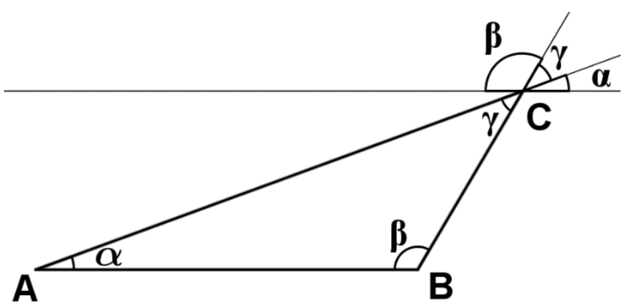




Draw radius OB, $|\text{OA}| = |\text{OB}| = |\text{OC}| = r$
We get two isosceles triangles with base angles α and β.
Assuming the sum of angles of a triangle to be 180° we get:
$$\alpha + \beta + (\alpha + \beta) = 180°$$
$$\alpha + \beta = 90° \quad \Rightarrow \quad \angle ABC = 90°$$

Show that sum of angles in a plane triangle is 180°.
This is not true for a triangle on a sphere so we need Euclidean geometry and the fifth axiom.



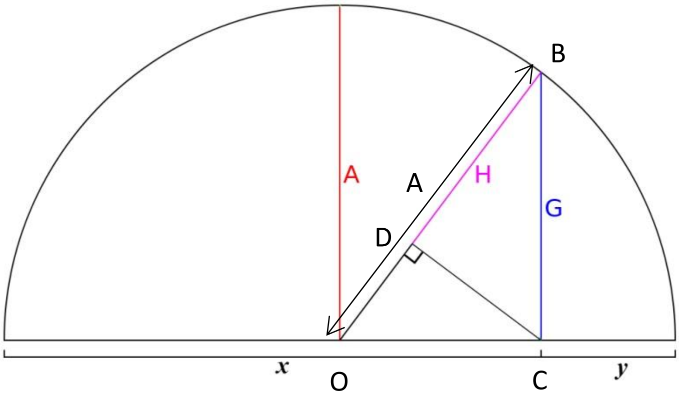


Draw a line through C parallel to AB.
Corresponding and opposing angles are equal.
$\alpha + \beta + \gamma$ corresponds to half a turn.

$$\alpha + \beta + \gamma = 180°$$

2.10 Show the inequalities $H \leq G \leq A$ among the Pythagorean means where

$$A = \frac{x+y}{2}, G = \sqrt{x \cdot y} \text{ and } H = \left(\frac{1/x+1/y}{2}\right)^{-1} \text{ with } x, y \in \mathbb{R}^+.$$



From the geometrical construction it is clear that
$\mathrm{H} \leq \mathrm{G}$
Since G is the hypotenuse in a triangle with side H.
$\mathrm{G} \leq \mathrm{A}$
Since the diameter is longer than a chord of a circle.

The geometrical construction is correct since:

A is constructed from the bisection of a segment of length $x + y$ is clearly equal to $(x + y)/2$.

$\triangle$OBC gives:

$$\mathrm{G}^2 + \left(x - \frac{x+y}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2$$
$$\mathrm{G}^2 = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$$
$$\mathrm{G} = \sqrt{x \cdot y}$$

$\triangle\mathrm{OBC} \sim \triangle\mathrm{BCD}$ (Similar triangles, one is a translated, scaled and in this case reflected version of the other)

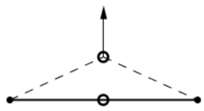
$$\frac{\mathrm{A}}{\mathrm{G}} = \frac{G}{H} \implies \mathrm{H} = \frac{\mathrm{G}^2}{\mathrm{A}} = \frac{xy}{(x+y)/2} \implies \mathrm{H}^{-1} = \frac{x+y}{2xy} \implies \mathrm{H} = \left(\frac{1/x + 1/y}{2}\right)^{-1}$$

From the construction it is also clear that:
$\mathrm{A} = \mathrm{G}$ if and only if $x = y$
$\mathrm{G} = \mathrm{H}$ if and only if $x = y$

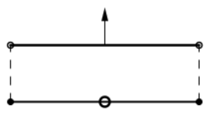
2.11 Describe the three regular convex n-polytopes of each dimension ≥ 5.

Assuming as stated in the text on polygons and polyhedra that there are only three regular convex n-polytopes for each dimension $n \geq 5$ we only need to find three examples in each dimension. We can do this by induction, starting with $n = 1$ and the unit segment.

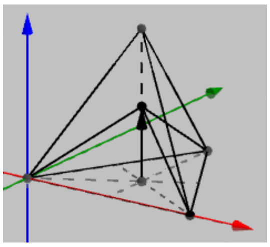
Case I $(n = 2)$
Locate a point at the center by bisection and move it perpendicularly into the next dimension until all vertices are equally separated to get an equi-lateral triangle aka a regular 2-simplex with 3 vertices (V), 3 edges (E) and 1 face (F).



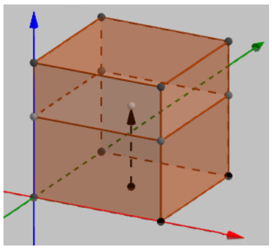
Case II $(n = 2)$
As above but copy the segment and sweep it along one unit to get a square with V=4, E=4, F=1.



Case I $(n = 3)$
Along the axis used used when n=2 there is a point where the distance to all three vertices are equal. Extend this point into a 3rd dimension until all four vertices are at equal distances. The convex hull of the vertices is a volume or cell(C) swept out by a triangle shrinking to a point. It is a regular tetrahedron also known as a regular 3-simplex with V=4, E=6, F=4, C=1.



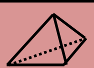
Case II $(n = 3)$
Halfway along the axis used when n=2 there is a point whith distances to all four vertices equal. Copy the square and move it one unit in a 3rd dimension. It will sweep out a cube with V=8, E=12, F=6, C=1.



The same procedures works iteratively to get n-dimensional n-polytopes. The case I n-polytope is a regular n-simplex. The case II n-polytope is a hypercube or n-cube, alternative names are tesseract, penteract etc. To get the number of elements in each case the procedures give iterative formulas for $e_{m,n}$ the number of m-dimenional elements in n dimensions.

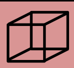
Dimension	Term
0	Vertex
1	Edge
2	Face
3	Cell
…	…
n-1	Facet
n	Body

Case I, n-simplices
$e_{m,n} = e_{m,n-1} + e_{m-1,n-1}$ with $e_{0,n} = n + 1$ and $e_{m,0} = 0$ for $m > 0$

n-simplex	*m* / *n*	*Vertex 0*	*Edge 1*	*Face 2*	*Cell 3*	*$e_{4,n}$ 4*
•	*0*	1	0	0	0	0
——	*1*	2	1	0	0	0
△	*2*	3	3	1	0	0
	*3*	4	6	4	1	0
5-cell	*4*	5	10	10	5	1
etc	*n*	n+1	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$

$$\binom{A}{B} = \frac{A!}{B!\,(A-B)!}$$
The number of ways to choose B objects from a collection of A objects.

Case II, n-cubes
$e_{m,n} = 2e_{m,n-1} + e_{m-1,n-1}$ with $e_{0,n} = 2^n$ and $e_{m,0} = 0$ for $m > 0$

n-cube	*m* / *n*	*Vertex 0*	*Edge 1*	*Face 2*	*Cell 3*	*$e_{4,n}$ 4*
•	*0*	1	0	0	0	0
——	*1*	2	1	0	0	0
□	*2*	4	4	1	0	0
	*3*	8	12	6	1	0
tesseract	*4*	16	32	24	8	1
etc	*n*	2^n	$\binom{n}{1}2^{n-1}$	$\binom{n}{2}2^{n-2}$	$\binom{n}{3}2^{n-3}$	$\binom{n}{4}2^{n-4}$

The terms $e_{m,n}$ in case I follow the pattern of Pascals triangle and case II elements resemble binomial coefficients used when expanding $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Case I, comparing with expansion of $(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k}$ gives:

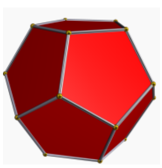
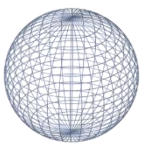
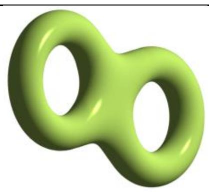
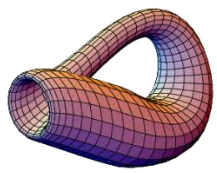
$$\begin{cases} n \text{ even:} & 0 = 1 - \mathrm{V} + \mathrm{E} - \mathrm{F} + \cdots + B \\ n \text{ odd:} & 0 = -1 + V - E + F - \cdots + B \end{cases} \Rightarrow \mathrm{V} - \mathrm{E} + \mathrm{F} - \mathrm{C} + \cdots \pm \mathrm{B} = 1$$

Case II, comparing with expansion $(1-2)^n = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k}$ gives

$$\begin{cases} n \text{ even:} & 1 = V - \mathrm{E} + \mathrm{F} + \cdots + B \\ n \text{ odd:} & -1 = -V + E - F + \cdots + B \end{cases} \Rightarrow \mathrm{V} - \mathrm{E} + \mathrm{F} - \mathrm{C} + \cdots \pm \mathrm{B} = 1$$

In two dimensions with $\mathrm{B} = \mathrm{F} = 1$: $\quad \mathrm{V} - \mathrm{E} = 0$
In three dimensions with $\mathrm{B} = \mathrm{C} = 1$: $\quad \mathrm{V} - \mathrm{E} + \mathrm{F} = 2$

$\chi = V - E + F$ is called the Euler characteristic for a polyhedron.
It can be generalized to surfaces with general topologies of lower and higher dimensions, both orientable and non-orientables objects. Each geometrical type has its own value of χ.

				
$\chi = 2$	$\chi = 2$	$\chi = 0$	$\chi = -2$	$\chi = 0$

The third regular convex n-polytope, the missing case III is the dual of case II with a vertex in the centre of each n-cube facet. The vertex coordinates are ± 1 along each cordinate axis, like a cross, the body is the convex hull of these vertices and their names are cross-polytopes.

2.12 Express the fraction $\frac{100\,000}{101\,001}$ from chapter one as a continued fraction and show that $[1;1,1,\dots] = 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}$ equals the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

Euclid's algorithm to find GCD(100 000,101 001) gives:

$$\begin{array}{rcll}
100\,000 & = & 0\cdot 101\,001 + 100\,000 & q_0 = 0\\
101\,001 & = & 1\cdot 100\,000 + 1\,001 & q_1 = 1\\
100\,000 & = & 99\cdot 1\,001 + 901 & q_2 = 99\\
1\,001 & = & 1\cdot 901 + 100 & q_3 = 1\\
901 & = & 9\cdot 100 + 1 & q_4 = 9\\
100 & = & 100\cdot 1 + 0 & q_5 = 100
\end{array}$$

$$\frac{100\,000}{101\,001} = \cfrac{1}{1+\cfrac{1}{99+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{100}}}}} = [0;1,99,1,9,100]$$

$1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}$ is defined as the limit of convergents $\{c_k\}_{k\in\mathbb{N}}$ (truncated continued fractions)

$$c_n = a_0 + \cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_n}}}} = a_0 + (a_1 + (a_2 + (\dots(a_{n-1}+a_n{}^{-1})^{-1}\dots)^{-1})^{-1})^{-1} = a_0 + \underset{i=1}{\overset{n}{\mathrm{K}}}\frac{1}{a_i}$$
$$= a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_{n-1}+}\,\frac{1}{a_n} = [a_0;a_1,a_2,\dots,a_{n-1},a_n]$$

n	0	1	2	3	4
c_n	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$

Fibonacci? $c_n = \frac{F_{n+2}}{F_{n+1}}$ with $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

Proof by induction: $c_0 = F_2/F_1$. Assume $c_n = F_{n+2}/F_{n+1}$

$$c_{n+1} = 1 + \underbrace{\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{1+}\,\frac{1}{1}}_{n+1} = 1 + \frac{1}{1+(c_n-1)} = 1 + \frac{F_{n+1}}{F_{n+2}} = \frac{F_{n+1}+F_{n+2}}{F_{n+2}} = \frac{F_{n+3}}{F_{n+2}}$$

n	0	1	2	3	4
$c_{n+1}-c_n$	$\frac{1}{1}$	$-\frac{1}{2}$	$\frac{1}{2\cdot 3}$	$-\frac{1}{3\cdot 5}$	$\frac{1}{5\cdot 8}$

Looks like: $c_{n+1} - c_n = \frac{(-1)^n}{F_{n+1}\cdot F_{n+2}}$

Proof by induction: $c_1 - c_0 = \frac{1}{F_1F_2}$. Assume $c_{n+1} - c_n = \frac{(-1)^n}{F_{n+1}\cdot F_{n+2}}$

$$c_{n+2} - c_{n+1} = \frac{F_{n+4}}{F_{n+3}} - \frac{F_{n+3}}{F_{n+2}} = \frac{F_{n+2}}{F_{n+3}} - \frac{F_{n+1}}{F_{n+2}} = \frac{1}{c_{n+1}} - \frac{1}{c_n} = \frac{c_n - c_{n+1}}{c_n\cdot c_{n+1}} = \frac{(-1)^{n+1}/(F_{n+1}F_{n+2})}{F_{n+3}/F_{n+1}} = \frac{(-1)^{n+1}}{F_{n+2}F_{n+3}}$$

$c_1 > c_3 > c_5 > \cdots > c_4 > c_2 > c_0$, $\frac{1}{F_{n+2}F_{n+3}} \to 0$ as $n\to\infty$ $\implies$

$\{c_k\}_{k\in\mathbb{N}}$ will oscillate towards a well-defined limit, call it x.

x must satisfy:

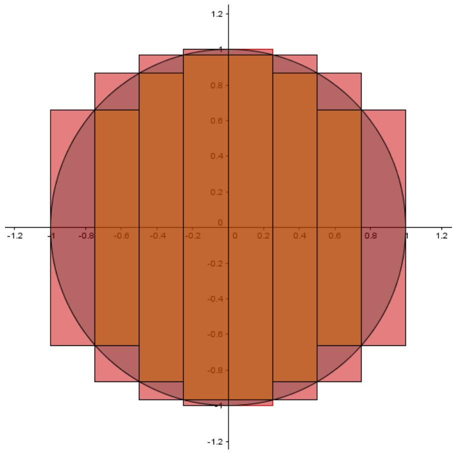
$$x = 1 + \frac{1}{x} \qquad\implies\qquad 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}} = \frac{1+\sqrt{5}}{2} = \varphi\ (\textit{the golden ratio})$$

$$x = \frac{1\pm\sqrt{5}}{2}$$

2.13 Derive the area of a disk of radius r by using rectangular decomposition.

Riemann integration gives the area using decomposition of the disk with rectangles. The lower limit of areas covering the disk and upper limit of areas inside the disk is shown to coincide. Using up-down and left-right symmetry of the disk we get:

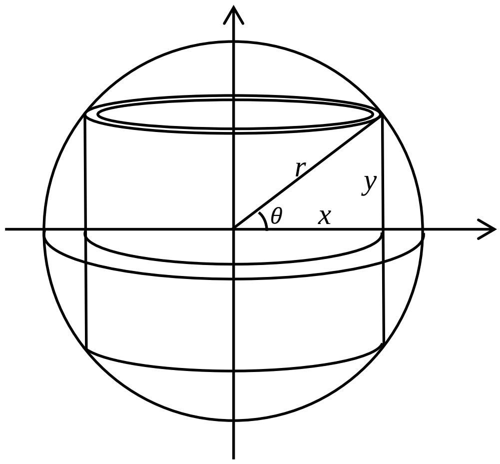
$$
\begin{aligned}
\text{Area} &= 4\int_0^r \sqrt{R^2 - x^2}\,dx \left[\begin{matrix} x = ru \\ \dfrac{dx}{du} = r \end{matrix}\right] \\
&= 4r^2 \int_0^1 \sqrt{1-u^2}\,du \left[\begin{matrix} u = \sin\theta \\ \dfrac{du}{d\theta} = \cos\theta \end{matrix}\right] \\
&= 4r^2 \int_0^{\pi/2} \cos^2\theta\, d\theta \\
&= 4r^2 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta \\
&= 4r^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} \\
&= \pi r^2
\end{aligned}
$$



Circle of radius R
$x^2 + y^2 = R^2$
$y = \sqrt{R^2 - x^2}$

The Riemann integral with variable substitution and primitive functions of trigonometric functions is presented in chapter three and there is no explicit or implicit use of what we set out to prove $A = \pi r^2$.

2.14 Derive the volume of a sphere.



Integrating over surface area of open cylinders:

Surface area: $2x \cdot \pi \cdot 2y$

$$\text{Volume of sphere } = 4\pi \int_0^r xy\,dx \left[\begin{array}{c} x = r\cos\theta \\ y = r\sin\theta \\ \frac{dx}{d\theta} = -r\sin\theta \end{array}\right]$$

$$\frac{4\pi r^3}{3}$$

$$= 4\pi r^3 \int_0^{\pi/2} (\cos\theta - \cos^3\theta)\,d\theta$$

$$= \pi r^3 \int_0^{\pi/2} (\cos\theta - \cos 3\theta)\,d\theta$$

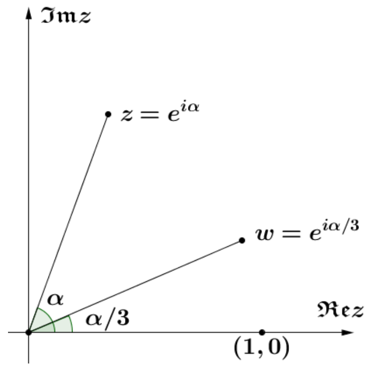
$$= 4\pi r^3/3$$

$$\boxed{\cos 3\theta = \mathrm{Re}\left(e^{3\theta i}\right) = \mathrm{Re}(\cos\theta + i\sin\theta)^3 = 4\cos^3\theta - 3\cos\theta \quad \Longrightarrow \cos^3\theta = (\cos 3\theta + 3\cos\theta)/4}$$

$$4\cos^3(\alpha/3) - 3\cos(\alpha/3) = \cos\alpha$$

Trisecting a general angle α requires a solution to a cubic equation, but ruler-and-compass construction starting from given points in the complex plane can only generate points expressible with $\left(+, -, \times, \div, \bar{z}, \sqrt{z}\right)$ operating on the original points.

The general cubic solution needs the operator $z^{1/3}$ which is not constructible from operators in the list.



$\cos(\alpha/3)$, $2^{1/3}$ and $\sqrt{\pi}$ are not constructible.

$\Downarrow$

Trisecting an angle, doubling a cube or squaring a circle is impossible. The ancient Greeks tried but failed.

Example of constructible number:

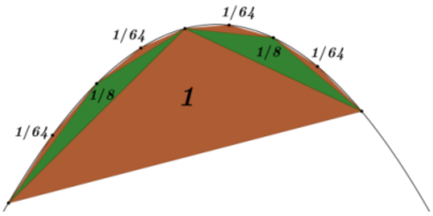
$$16\cos\left(\frac{2\pi}{17}\right) = -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

$\Longrightarrow$ Regular heptadecagon (17-gon) is constructible with ruler and compass.

(Proved by Carl Friedrich Gauss in 1796 at age 17)

2.15 Show that the area of a parabolic segment can be seen as a sum of areas of inscribed triangles that form a geometric series.

$$A = A_1 \cdot \sum_{i=1}^{\infty} \frac{2^i}{8^i} = \frac{4}{3} A_1$$



A general parabola can be described as $y = k(x-\alpha)^2 + \beta$ in a coordinate system (with z-axis). Translation and reflection (in x-axis) will not change areas. We can assume $\alpha, \beta = 0\,,\ k > 0$. We will consider ratios of areas of inscribed triangles.
A ratio of areas is unaffected by a stretch in the y-direction, assume $k = 1$.

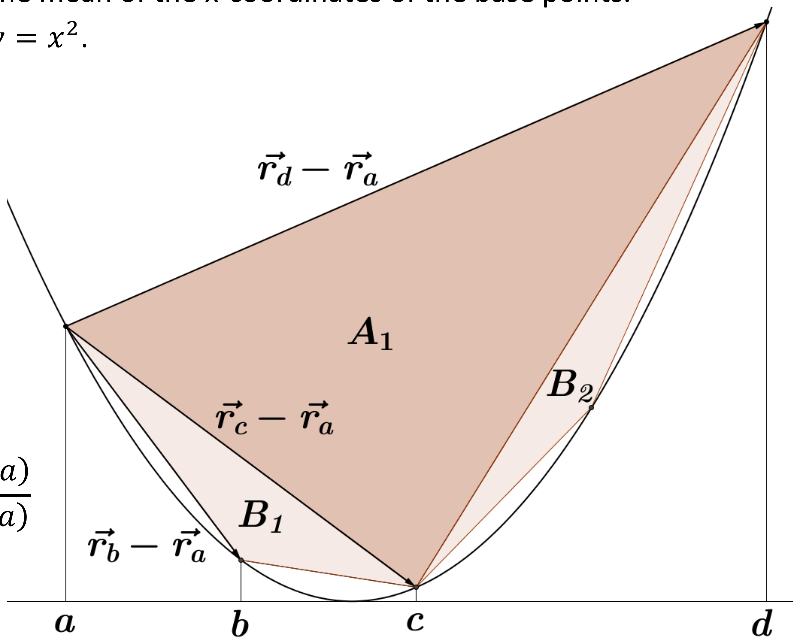
Each new inscribed triangle is formed with its base on an existing one.
Let the x-coordinate of the new point be the mean of the x-coordinates of the base points.
The y-coordinate will lie on the parabola $y = x^2$.

$c = \dfrac{a+d}{2}$ and $b = \dfrac{a+c}{2}$

$\vec{r_x} = (x, x^2, 0)$

$2A_1 = |(\vec{r_c} - \vec{r_a}) \times (\vec{r_d} - \vec{r_a})|$
$2B_1 = |(\vec{r_b} - \vec{r_a}) \times (\vec{r_c} - \vec{r_a})|$

$$\frac{A_1}{B_1} = \frac{(c-a)(d^2-a^2) - (c^2-a^2)(d-a)}{(b-a)(c^2-a^2) - (b^2-a^2)(c-a)} = \frac{(c-a)(d-a)(d-c)}{(b-a)(c-a)(c-b)} = 8$$



Similarly $A_1/B_1 = 8$, $A_1/C_1 = 8^2$ etc.
so $B_1 + B_2 = \frac{1}{4} A_1$ and $C_1 + C_2 + C_3 + C_4 = \frac{1}{4^2} A_1$ etc.

Area of a parabolic segment with A_1 = area of inscribed triangle with chord and apex at horizontal mean:

$$A = A_1 + (B_1 + B_2) + (C_1 + C_2 + C_3 + C_4) + \cdots = A_1 \cdot \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{4}{3} A_1$$

2.16 Solve the cattle problem of Archimedes:

" Compute, O friend the number of cattle of the sun which once grazed upon the plains of Sicily, divided according to color into four herds, …"
They were white, yellow, black and dappled, bulls (W, Y, B, D), cows (w, y, b, d).
There were more bulls than cows and their numbers were as:

$$\begin{aligned} W &= \left(\tfrac{1}{2}+\tfrac{1}{3}\right)B+Y \\ B &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right)D+Y \\ D &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right)W+Y \end{aligned} \qquad \begin{aligned} w &= \left(\tfrac{1}{3}+\tfrac{1}{4}\right)(B+b) \\ b &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right)(D+d) \\ d &= \left(\tfrac{1}{5}+\tfrac{1}{6}\right)(Y+y) \\ y &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right)(W+w) \end{aligned}$$

$W+B$ is a square number
$D+Y$ a triangular number

Find the number of cattle which once grazed upon the plains of Sicily.

The first part is a linear problem to find $\mathbf{v} = (W, B, D, Y, w, b, d, y)^T \in \mathbb{Z}^{\mathbf{8}}$ with $\mathbf{m} \cdot \mathbf{v} = \mathbf{0}$.

$$\mathbf{m} = \begin{pmatrix} -1 & \frac{5}{6} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & \frac{9}{20} & 1 & 0 & 0 & 0 & 0 \\ \frac{13}{42} & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{12} & 0 & 0 & -1 & \frac{7}{12} & 0 & 0 \\ 0 & 0 & \frac{9}{20} & 0 & 0 & -1 & \frac{9}{20} & 0 \\ 0 & 0 & 0 & \frac{11}{30} & 0 & 0 & -1 & \frac{11}{30} \\ \frac{13}{42} & 0 & 0 & 0 & \frac{13}{42} & 0 & 0 & -1 \end{pmatrix}$$

Using Mathematica and NullSpace$[\mathbf{m}]$ gives the vectors spanning the solution space.

$$\mathbf{v} = \alpha\left(\frac{3455494}{1813071}, \frac{828946}{604357}, \frac{7358060}{5439213}, \frac{461043}{604357}, \frac{2402120}{1813071}, \frac{543694}{604357}, \frac{1171940}{1813071}, 1\right)^{\mathrm{T}} \quad \alpha \in \mathbb{R}$$

$\frac{\text{Bulls}}{\text{Cows}} = \frac{W+B+D+Y}{w+b+d+y} \approx 1.39$ in the spanning vector, more bulls than cows is satisfied.
Multiplying with the least common multiple (LCM) of the denominators, $(\mathrm{LCM} = 3^2 \cdot 13 \cdot 46489)$.

$$\begin{aligned} W &= 10\,366\,482 \cdot n & w &= 7\,206\,360 \cdot n \\ B &= 7\,460\,514 \cdot n & b &= 4\,893\,246 \cdot n \\ D &= 7\,358\,060 \cdot n & d &= 3\,515\,820 \cdot n \qquad n \in \mathbb{Z}^+ \\ Y &= 4\,149\,387 \cdot n & y &= 5\,439\,213 \cdot n \end{aligned}$$

$W+B$ square number, all primefactors have even exponents:
$W + B = 2^2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657 \cdot n \Longrightarrow n = 3 \cdot 11 \cdot 29 \cdot 4657 \cdot q^2\,,\ \ q \in \mathbb{Z}$

$D + Y$ triangle number of form $N(N+1)/2$

$D + Y = kq^2$

$k = (7358060 + 4149387) \cdot 3 \cdot 11 \cdot 29 \cdot 4657$

$kq^2 = \dfrac{N(N+1)}{2} \quad q, N \in \mathbb{Z}^+$

$N = \dfrac{\sqrt{8kq^2+1} - 1}{2} \implies$

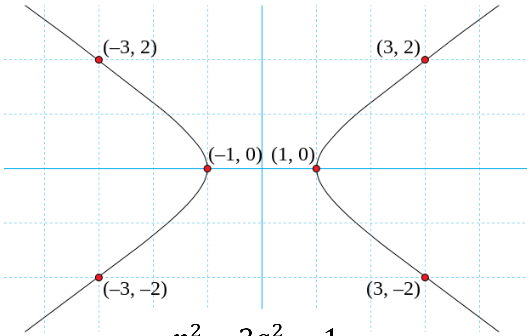
$8kq^2 + 1 = p^2 \quad q \in \mathbb{Z}^+,\ p \in 2\mathbb{Z}^+ + 1$

$p^2 - Kq^2 = 1 \quad K =$410 286 423 278 424

$$p \in 2\mathbb{Z} \Rightarrow p^2 - Kq^2 \in 2\mathbb{Z} \not\ni 1$$

$$\Downarrow$$

Problem reduced to finding $(p, q) \in (\mathbb{Z}^+, \mathbb{Z}^+)$ on a hyperbola $p^2 - Kq^2 = 1$ (Pell's equation)



$p^2 - 2q^2 = 1$

Mathematica code to find solution to cattle problem:

```
W=10366482;B=7460514;D=7358060;Y=4149387;
w=7206360;b=4893246;d=3515820;y=5439213;
Total1=W+B+D+Y+w+b+d+y;
FactorInteger[W+B]
  {{2,2},{3,1},{11,1},{29,1},{4657,1}}
m=3·11·29·4657;
k=(D+Y) ·m;
K=8k;
pqRule=FindInstance[p^2-K·q^2,{p,q},Integers];
Total2=Total1·m·q^2 /. pqRule[[1]];
N[Total2,10]
```

$7.760271406 \cdot 10^{206544}$

The total number of cattle is:

$\underbrace{7760271406 \ldots 455081800}_{206545 \text{ digits}} \cdot n$

The problem was discovered in 1769
and solved by A. Amthor in 1880.
All digits were first printed in 1965.

To find solutions $(p, q) \in (\mathbb{Z}., \mathbb{Z})$ to Pell's equation, start from $p^2 - Kq^2 = 1 \Rightarrow p/q = \sqrt{K + 1/q^2}$. The theory was developed by Lagrange in 1766–1769.

If K is a square natural number the problem will be trivial since
$p^2 - Kq^2 = (p + kq)(p - kq)$ with $k = \sqrt{K} \in \mathbb{Z}^+$.

A non-square K has a palindromic periodic continued fraction expansion
$\sqrt{K} = \left[\lfloor\sqrt{K}\rfloor; \overline{a_1, a_2, \ldots, a_2, a_1, 2\lfloor\sqrt{K}\rfloor}\right]$ with a repeating period of length r.

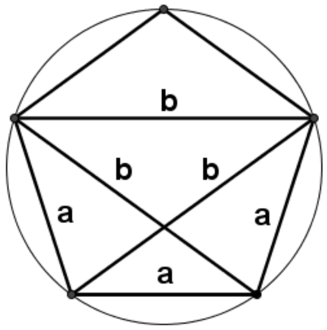
Pell's equation has a non-trivial solution (p, q), the positive one with minimal p is called the minimal solution (p_1, q_1). It is found among the convergents $h_n/k_n \equiv [x_0; x_1, \ldots, x_n]$ of $\sqrt{K}$.

$$(p_1, q_1) = \begin{cases} (h_{r-1}, k_{r-1}) & \text{if } r \text{ is even} \\ (h_{2r-1}, k_{2r-1}) & \text{if } r \text{ is odd} \end{cases}$$

All solutions can be derived from symmetry (reflection in x/y-axis) and
from powers of the minimal solution: $p + q\sqrt{K} = \pm(p_1 + q_1\sqrt{K})^{\pm n}$, $n \in \{0,1,2,\ldots\}$
An effort was made in 1867 to solve the Cattle problem but $\sqrt{410\,286\,423\,278\,424}$ has $r = 203254$
Amthor solved it by removing a square factor $2^2 \cdot 4657^2$ to get a square-free root with $r = 92$.

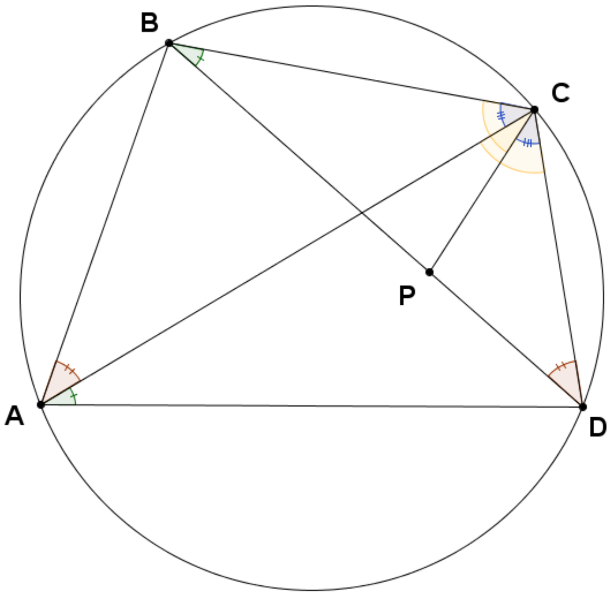
If Archimedes' cattle problem was really formulated by Archimedes is uncertain. It was discovered in an old Greek manuscript in a German library and published in 1773. There was plenty of time from Archimedes death in 212 BC for somebody with an interesting math problem and some knowledge of antiquity and Greek to attribute it to Archimedes in a nicely formulated letter to Eratosthenes, the librarian at Alexandria. The problem is not mentioned in any known sources of ancient Greek origin.

2.17 Show that the ratio of the diagonal to the side in a regular pentagon equals the golden ratio, $\frac{b}{a} = \varphi \equiv \frac{1+\sqrt{5}}{2}$.



Ptolemy's theorem on the circular quadrilateral gives:

$$b^2 = ba + a^2$$
$$\left(\frac{b}{a}\right)^2 = \frac{b}{a} + 1$$
$$\frac{b}{a} = \frac{1+\sqrt{5}}{2}$$



Ptolemy's theorem:

In a circular quadrilateral ABCD: AC·BD=AB·CD+AD·BC

Put a point P on line BD such that $\angle BCA = \angle PCD$

$$\Delta ABC \sim \Delta DPC \;\Rightarrow \frac{AC}{AB} = \frac{DC}{DP} \Rightarrow AC \cdot DP = AB \cdot DC$$

$$\Delta ACD \sim \Delta BCP \;\Rightarrow \frac{AC}{AD} = \frac{BC}{BP} \Rightarrow AC \cdot BP = AD \cdot BC$$

$$AC \cdot (DP + BP) = AB \cdot DC + AD \cdot BC$$

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

3.1 Show that a logical n-ary operator $Q(P_1, \ldots, P_n)$ with a specified truth table can be given by a formula based on P_i , $\neg$ and $\wedge$.

	P_1	P_2	$\cdots$	P_{n-1}	P_n	Q
$(2^n-1)_2$	1	1	$\cdots$	1	1	T_1
$(2^n-2)_2$	1	1	$\cdots$	1	0	T_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$(1)_2$	0	0	$\cdots$	0	1	T_{2^n-1}
$(0)_2$	0	0	$\cdots$	0	0	T_{2^n}

$T_i \in \{0,1\}$

Table of 1s and 0s below row of Ps
T_{ij}
$1 \le i \le 2^n$
$1 \le j \le n$

To make a row with $T_i = 1$ true we use AND on every column with $\begin{cases} P_j \text{ if 1 in column} \\ \neg P_j \text{ if 0 in column} \end{cases}$.

Combine these formulas with OR and every row with $T_i = 1$ will be correct once and therefore true. No row with $T_i = 0$ will be included and therefore false for each argument of OR and so false.

$$Q = \bigvee_{\{i: T_i = 1\}} \left(\bigwedge_{j=1}^{n} \begin{cases} P_j \text{ if } \mathrm{T_{ij}} = 1 \\ \neg P_j \text{ if } \mathrm{T_{ij}} = 0 \end{cases} \right)$$

Since A ∨ B can be replaced with $\neg(\neg A \wedge \neg B)$ there is no need for ∨, only P_i , $\neg$ and $\wedge$.

Example:

P_1	P_2	$Q = P_1 \to P_2$
1	1	1
1	0	0
0	1	1
0	0	1

$Q = (P_1 \wedge P_2) \vee (\neg P_1 \wedge P_2) \vee (\neg P_1 \wedge \neg P_2)$

Boolean algebra:

Use distributive law to get combination of P_i with itself.

$= \underbrace{(P_1 \vee \neg P_1 \vee \neg P_1)}_{1} \wedge \underbrace{(P_1 \vee \neg P_1 \vee \neg P_2)}_{1}$

$\wedge \underbrace{(P_1 \vee P_2 \vee \neg P_1)}_{1} \wedge \underbrace{(P_1 \vee P_2 \vee \neg P_2)}_{1}$

$\wedge \underbrace{(P_2 \vee \neg P_1 \vee \neg P_1)}_{P_2 \vee \neg P_1} \wedge \underbrace{(P_2 \vee \neg P_1 \vee \neg P_2)}_{1}$

$\wedge \underbrace{(P_2 \vee P_2 \vee \neg P_1)}_{P_2 \vee \neg P_1} \wedge \underbrace{(P_2 \vee P_2 \vee \neg P_2)}_{1}$

$= P_2 \vee \neg P_1 = \neg(P_1 \wedge \neg P_2)$ (Obviously true from truth table)

Rules of Boolean algebra:

$x \vee (y \vee z)$	$=$	$(x \vee y) \vee z$	Associativity of ∨
$x \wedge (y \wedge z)$	$=$	$(x \wedge y) \wedge z$	Associativity of ∧
$x \vee y$	$=$	$y \vee x$	Commutativity of ∨
$x \wedge y$	$=$	$y \wedge x$	Commutativity of ∧
$x \wedge (y \vee z)$	$=$	$(x \wedge y) \vee (x \wedge z)$	Distributivity of ∧ over ∨
$x \vee (y \wedge z)$	$=$	$(x \vee y) \wedge (x \vee z)$	Distributivity of ∨ over ∧
$x \vee 0$	$=$	x	Identity for ∨
$x \wedge 1$	$=$	x	Identity for ∧
$x \vee 1$	$=$	1	Annihilator for ∨
$x \wedge 0$	$=$	0	Annihilator for ∧
$x \vee x$	$=$	x	Idempotence for ∨
$x \wedge x$	$=$	x	Idempotence for ∧
$x \vee \neg x$	$=$	1	Complementation for ∨
$x \wedge \neg x$	$=$	0	Complementation for ∧
$\neg(x \vee y)$	$=$	$\neg x \wedge \neg y$	De Morgan's law I
$\neg(x \wedge y)$	$=$	$\neg x \vee \neg y$	De Morgan'slaw II
$\neg\neg x$	$=$	x	Double negation

3.2 Conway's arrow notation $c_1 \to c_2 \to \cdots \to c_n$ is defined recursively:

1. $p \to q \equiv p^q \quad (p, q \in \mathbb{Z}^+)$
2. $X \to 1 \equiv X \quad$ (X is any chained expression)
3. $X \to p \to (q+1) \equiv \underbrace{X \to (X \to (\cdots (X \to (X) \to q) \cdots) \to q) \to q}_{p \text{ repetitions of } X}$

Knuth's up-arrow notation $a \uparrow^n b$ $(a, b, n \in \mathbb{Z}^+)$ is defined recursively as:

$a \uparrow b = a^b$

$a \uparrow^{n+1} b \equiv \underbrace{a \uparrow^n \left(a \uparrow^n (\ldots \uparrow^n a)\right)}_{b \text{ repetitions of } a}$

Show that:

- Conway chained arrow notation is not an iterated binary operator and
- $p \to q \to r = p \uparrow^r q$
- Express $3 \to 3 \to 3 \to 2$ in Knuth's up-arrow notation.

To show that chained arrow notation is not an iterated binary operator it will be enough to find a counterexample that shows $a \to b \to c \neq (a \to b) \to c$ and $\to b \to c \neq a \to (b \to c)$.

$(4 \to 2) \to 3 = (4^2)^3 = 4^6 = 4 \uparrow 6$

$4 \to (2 \to 3) = 4^{2^3} = 4^8 = 4 \uparrow 8$

$4 \to 2 \to 3 = 4 \to (4) \to 2 = 4 \to (4 \to (4 \to (4) \to 1) \to 1) \to 1 = 4^{4^{4^4}} = 4 \uparrow\uparrow 4 = 4 \uparrow^3 2$

r $= 1$: $p \to q \to 1 = p^q = p \uparrow^1 q$

Assume $p \to q \to r = p \uparrow^r q$ true for $r \leq n$.

$p \to q \to (n+1) = p \to (p \to (\cdots (p \to (p \to (p) \to n) \to n) \cdots) \to n \to n$ (q repetitions)

$= p \to (p \to (\cdots (p \to \quad (p \uparrow^n p) \quad \to n) \cdots) \to n \to n$

$= p \to (p \to (\cdots (\quad p \uparrow^n (p \uparrow^n p) \quad) \cdots) \to n \to n$

$= p \uparrow^n (p \uparrow^n (\cdots \quad (p \uparrow^n \ (p \uparrow^n p)) \cdots))$ (q repetitions)

$= p \uparrow^{n+1} q$

By induction $p \to q \to r = p \uparrow^r q$ for every p, q, r in $\mathbb{Z}^+$.

$\underbrace{3 \to 3}_{X} \to \underbrace{3}_{p} \to \underbrace{2}_{q+1} = 3 \to 3 \to (3 \to 3 \to (3 \to 3) \to 1) \to 1$

$= 3 \to 3 \to \quad \left(3 \to 3 \to \ (3 \uparrow 3)\right) \quad \to 1$

$= 3 \to 3 \to \quad \left(3 \uparrow^{(3\uparrow 3)} 3\right)$

$= 3 \uparrow^{\left(3\uparrow^{(3\uparrow 3)} 3\right)} 3$

$3 \to 3 \to 3 \to 2 = 3 \uparrow^{\left(3\uparrow^{(3\uparrow 3)} 3\right)} 3$

$3 \to 3 \to n \to 2$ will be stacked n levels

Introducing $f(n) = 3 \uparrow^n 3 = 3 \to 3 \to n$ from Big numbers part 2 in the book.

$f(1) = 27 \quad f^2(1) = f\left(f(1)\right) = 3 \to 3 \to (3 \to 3 \to 1)$

$f^n(1) = 3 \to 3 \to (3 \to 3 \to (\ldots (3 \to 3 \to 1) \ldots)$ n rep. $= 3 \to 3 \to n \to 2$

$3 \to 3 \to 3 \to 3 = 3 \to 3 \to (3 \to 3 \to 27 \to 2) \to 2 = f^{3\to 3\to 27\to 2}(1) = f^{3\to 3\to 27\to 2-1}(27)$

This number is quite a bit larger than Grahams number $G = f^{64}(4)$ from Big numbers part 2.

3.3 Show that a sum of powers of degree p is a polynomial of degree $p+1$ and derive the polynomial $S_p(n)$. Do this for $p=3$, $p=4$ and possibly beyond.

$$S_p(n) = \sum_{k=1}^{n} k^p$$

$p=3$: $S_3(n) = 1^3 + 2^3 + \dots + n^3 = \underbrace{a_0 n^0 + a_1 n^1 + a_2 n^2 + a_3 n^3 + a_4 n^4}_{P(n)}$

$S_3(0) = 0 \rightarrow a_0 = 0 \quad \left(\int_0^n x^3 dx < \sum_{k=0}^{n} k^3 < \int_1^{n+1} x^3 dx \rightarrow \text{we could assume } a_4 = \frac{1}{4}, \text{but will not}\right)$

n	$(1^3 + \dots + n^3)$	$P(n)$
1	1^3	$a_1 + a_2 + a_3 + a_4$
2	$1^3 + 2^3$	$2a_1 + 2^2a_2 + 2^3a_3 + 2^4a_4$
3	$1 + 2^3 + 3^3$	$3a_1 + 3^2a_2 + 3^3a_3 + 3^4a_4$
4	$1 + 2^3 + 3^3 + 4^3$	$4a_1 + 4^2a_2 + 4^3a_3 + 4^4a_4$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2^2 & 2^3 & 2^4 \\ 3 & 3^2 & 3^3 & 3^4 \\ 4 & 4^2 & 4^3 & 4^4 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 \\ 1+2^3 \\ 1+2^3+3^3 \\ 1+2^3+3^3+4^3 \end{pmatrix}}_{X}$$

Mathematica code:

```
M = Table[ i^j , {i,4} , {j,4} ]
X = Table [Sum[ i^3 , {i,1,j} ], {j,4}]
A = Inverse[m].x
```

Results:

$A[i] = \left(0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right) \rightarrow \bar{\bar{S}}_3(n) = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$

$S_3(n) = (1^3 + \dots + n^3) \rightarrow S_3(n) = (1,9,36,100,225,441, \dots)$

$\bar{\bar{S}}_3(n) = (1,9,36,100,225,441, \dots)$

$S_3(n) = \bar{\bar{S}}_3(n)$ is true for $n = 1,2,3,4$. Assume it is true for n and show $4S_3(n+1) = 4\bar{\bar{S}}_3(n+1)$:

$4S_3(n+1)$	$4\bar{\bar{S}}_3(n+1)$
$4(S_3(n) + (n+1)^3)$	$(n+1)^2 + 2(n+1)^3 + (n+1)^4$
$4\bar{\bar{S}}_3(n) + 4(1 + 3n + 3n^2 + n^3)$	$4 + 12n + 13n^2 + 6n^3 + n^4$
$4 + 12n + 13n^2 + 6n^3 + n^4$	

In the same way we find:

$S_0(n) = n$

$S_1(n) = \frac{1}{2}n^2 + \frac{1}{2}n$

$S_2(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$

$S_3(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$

$S_4(n) = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$S_5(n) = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$

$S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

$S_7(n) = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n$

Sums of powers have interested mathematicians since 500 BC. Pythagoras derived $S_1(n)$ and Archimedes derived $S_2(n)$. Aryabhata in India discovered the formula for $S_3(n)$ c. 500 AD. Abu Bakr al-Karaji, Bagdad gave a proof of $S_3(n)$ in c. 1000 AD and Abu ibn al-Haytham derived $S_4(n)$ in the same period. Johann Faulhaber, Germany calculated $S_p(n)$ for $p = 1, \dots, 17$ in c. 1600. Secret messages in exercises reveal that he reached $p = 25$ without publishing it. The general formula for $S_p(n)$ is named after him but he never proved it. The first proof was given by Carl Jacobi in 1834. To find the true formula we will index powers downwards and extract a factor of $1/(p+1)$.

$$S_p(n) = \frac{1}{p+1} \sum_{j=0}^{p} C(p,j) n^{p+1-j}$$

$C(p,j)$ seem to be easiest to guess column-wise, with $C(p,0) = 1$ and $C(p,1) = (p+1) \cdot 1/2$. Turning to denominator 12 in column three reveals $C(p,2) = (p+1) \cdot p \cdot 1/12$. No n^{p-3} terms means $C(p,3) = 0$. The next column starting with $-1/30$ seems harder to figure out, one guess is to continue the trend of a factor of falling factorials $C(p,4) = (p+1)p(p-1)(p-2) \cdot C$.

By making a list of M, Table[Table[i^j , {i,m+1} , {j,m+1}], {m,17}] and similarly for X, calculating $S_p(n)$ and extracting $C(p,j)$ it seems reasonable to assume that $C(p,j) = (p+1)p(p-1)\dots(p+2-j)A_j$ The general formula for sums of powers $S_p(n)$ is named in honor of Faulhaber. It is written:

$$S_p(n) = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j} \qquad (B_j)_0^{\infty} = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots\right)$$

The factor $(-1)^j$ can seem a bit confusing. It only matters for $j = 1$ since B_j is zero for every odd $j > 1$. The constants B_j are called Bernoulli numbers, they come in two versions that only differ for B_1.The first version has $B_1 = -½$ and the second version has $B_1 = +½$. This difference corresponds to $+n^2/2$ and $-n^2/2$ in $S_p(n)$, the same difference occurs for a slightly different convention for $S_p(n)$ that start with 0^p and ends with $(n-1)^p$. The Bernoulli numbers have deep connections to number theory. They are named after Jakob Bernoulli (1655—1705) who discovered that the coefficient of n^{m-j} is always a constant times $m!/(m-j)!$. The following proof of the general formula is inspired by “Concrete Mathematics” written by Graham, Knuth and Patashnik. It is a proof by induction and the goal will be to find the values of B_j that will make the proof work.

$$S_p(n) = \sum_{k=0}^{n-1} k^p \quad \bar{\bar{S}}_p(n) = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j}B_j n^{p+1-j} \quad \text{Show } S_p(n) = \bar{\bar{S}}_p(n)$$

$$p = 0\colon\ S_0(n) = \sum_{k=0}^{n-1} k^0 = n \quad (0^0 = 1 \text{ has been assumed. It is the most natural convention})$$

$$\bar{\bar{S}}_0(n) = B_0 n \qquad \rightarrow \qquad S_0(n) = \bar{\bar{S}}_0(n) \text{ if } B_0 = 1$$

Assume $S_i(n) = \bar{\bar{S}}_i(n)$ for $0 \le i < p$.

$$S_{p+1}(n) + n^{p+1} = \sum_{k=0}^{n-1}(k+1)^{p+1} = \sum_{k=0}^{n-1}\sum_{j=0}^{p+1}\binom{p+1}{j}k^j = \sum_{j=0}^{p+1}\binom{p+1}{j}S_j \rightarrow$$

$$\begin{aligned}
n^{p+1} &= \sum_{j=0}^{p}\binom{p+1}{j}S_j(n) \\
&= \sum_{j=0}^{p}\binom{p+1}{j}\bar{\bar{S}}_j(n) + \binom{p+1}{p}S_p(n) - \binom{p+1}{p}\bar{\bar{S}}_p(n) \qquad \text{By assumption of induction} \\
&= \sum_{j=0}^{p}\binom{p+1}{j}\bar{\bar{S}}_j(n) + (p+1)\left(\underbrace{S_p(n) - \bar{\bar{S}}_p(n)}_{\Delta}\right) \\
&= \sum_{j=0}^{p}\binom{p+1}{j}\frac{1}{j+1}\sum_{k=0}^{j}\binom{j+1}{k}B_k n^{j+1-k} + (p+1)\Delta \qquad \text{By definition of } \bar{\bar{S}}_j(n)
\end{aligned}$$

Our aim will be to show $\Delta = 0$ for a suitable choice of B_j.

$$\begin{aligned}
n^{p+1} &= \sum_{0\le k\le j\le p} \binom{p+1}{j}\binom{j+1}{k}\frac{B_k}{j+1}n^{j+1-k} + (p+1)\Delta \\
&= \sum_{0\le k\le j\le p} \binom{p+1}{j}\binom{j+1}{j-k}\frac{B_{j-k}}{j+1}n^{k+1} + (p+1)\Delta \qquad \text{k} \to \text{j} - \text{k}, \text{terms permuted} \\
&= \sum_{0\le k\le j\le p} \binom{p+1}{j}\binom{j+1}{k+1}\frac{B_{j-k}}{j+1}n^{k+1} + (p+1)\Delta \qquad \binom{a}{b} = \binom{a}{a-b} \\
&= \sum_{0\le k\le p} \frac{n^{k+1}}{k+1} \sum_{k\le j\le p} \binom{p+1}{j}\binom{j}{k} B_{j-k} + (p+1)\Delta \\
&= \sum_{0\le k\le p} \frac{n^{k+1}}{k+1}\binom{p+1}{k} \sum_{k\le j\le p} \binom{p+1-k}{j-k} B_{j-k} + (p+1)\Delta \qquad \binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c} \\
&= \sum_{0\le k\le p} \frac{n^{k+1}}{k+1}\binom{p+1}{k} \sum_{0\le j\le p-k} \binom{p+1-k}{j} B_j + (p+1)\Delta \qquad \text{changed index } j-k \to j \\
&= \sum_{0\le k\le p} \frac{n^{k+1}}{k+1}\binom{p+1}{k} \delta_{pk} + (p+1)\Delta \qquad \text{If} \sum_{0\le j\le p-k} \binom{p+1-k}{j} B_j = \delta_{pk} \\
&= \frac{n^{p+1}}{p+1}\binom{p+1}{p} + (p+1)\Delta \\
&= n^{p+1} + (p+1)\Delta \qquad \Rightarrow \qquad \Delta = 0 \qquad \Rightarrow \qquad S_p(n) = \bar{\bar{S}}_p(n)
\end{aligned}$$

If B_j can be chosen s.t. $\sum_{0\le j\le N} \binom{N+1}{j} B_j = \delta_{N,0}$ for every $N \in \mathbb{N}_0$ then $S_p(n) = \bar{\bar{S}}_p(n)$ for $n \in \mathbb{N}_0$

$$\begin{array}{lll}
N = 0 & \binom{1}{0}B_0 = 1 \to & B_0 = 1 \\
N = 1 & \binom{2}{0}B_0 + \binom{2}{1}B_1 = 0 \to & B_1 = -\frac{1}{2} \\
N = 2 & \binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 = 0 \to & B_2 = \frac{1}{6} \quad \text{etc.}
\end{array}$$

These numbers called Bernoulli numbers were discovered independently by Jakob Bernoulli from Switzerland and Seki Kowa from Japan. Both their works on these numbers were published posthumously, Seki's in 1712 and Bernoulli's in 1713.

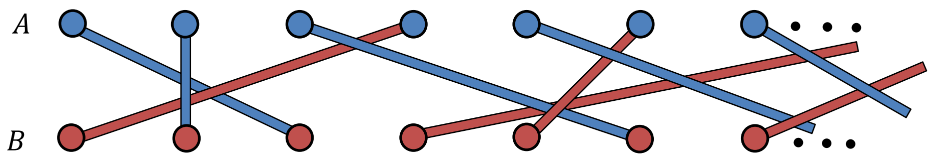
$$(B_j)_0^\infty = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, -\frac{3617}{510}, 0, \frac{43867}{798}, 0, \frac{174611}{330}, \ldots\right)$$

$$\boxed{0^p + 1^p + 2^p + \cdots + (n-1)^p = \frac{1}{p+1}\sum_{j=0}^{p}\binom{p+1}{j} B_j n^{p+1-j}}$$

3.4 Prove that if two sets are countable, totally ordered, dense and without upper and lower bounds then they are order-isomorphic.

Let $(A, \leq_A)$ and $(B, \leq_B)$ be two such ordered sets, $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$.
The conditions make them countably infinite.
Define a bijective function $f: A \to B: a_i \mapsto f(a_i) = b_{g(i)}$ by repeatedly following rule 1 and 2 :

1. Let i be the smallest index i s.t. a_i is not yet paired with a member of B.
   If a_i is strictly smaller than any of the previously paired elements of A,
     pair it with an element of B strictly smaller than any previously paired element of B.
   Elseif a_i is strictly larger than any of the previously paired elements of A,
     pair it with an element of B strictly larger than any previously paired element of B.
   Else $a_x < a_i < a_y$ where a_x is the largest of paired elements of A smaller than a_i
     and vice versa for a_y. Pair a_i with an element b_j s.t. $f(a_x) < b_j < f(a_y)$.
2. Let j be the smallest index j s.t. b_j is not yet paired with a member of A.
   Repeat the conditions under rule 1 word for word with $A \leftrightarrow B, a \leftrightarrow b$ and $i \leftrightarrow j$.



Total order guarantees that all elements in each set A and B are comparable with each other.
Denseness and lack of upper and lower bounds guarantee that mates to a_i and b_j can be found.
Countability guarantees that every element in A and B is in a list and will eventually be paired of.
f is a bijection and the construction of f guarantees that $a_m \leq_A a_n \Longrightarrow f(a_m) \leq_B f(a_n)$ so $(A, \leq_A)$ and $(B, \leq_B)$ are order-isomorphic.

The statement was first proved by Georg Cantor in 1895 by other methods. The method presented above is called the back-and-forth method. In all its simplicity it is of a surprisingly late date.
It was introduced by Huntington and Hausdorff in the beginning of the 20th century.

3.5 Exercises on cardinality of sets:

a) Show that $|\mathbb{R}| = |(0,1)|$.

b) Show that $|\mathcal{P}(A)| > |A|$ for any set A.

c) Show $|A| \leq |B|$ and $|B| \leq |A| \Rightarrow |A| = |B|$. (The Schröder-Bernstein theorem)

d) Find a bijective function $h: [0,1] \rightarrow (0,1)$.

---

a) $f: (0,1) \rightarrow \mathbb{R}\,, x \mapsto \frac{x-1/2}{x(1-x)}$ is continuous for $0 < x < 1$ with

$f'(x) = \frac{(x-1/2)^2+1/2}{x^2(x-1)^2} > 0 \Longrightarrow f$ is strictly increasing $\Longrightarrow f$ is injective.

$\lim_{x\to 0+} f(x) = -\infty$ and $\lim_{x\to 1-} f(x) = +\infty$.

The intermediate value theorem for continuous functions $\Longrightarrow f$ is surjective.

$f: (0,1) \rightarrow \mathbb{R}$ is bijective $\Longrightarrow \{x \in \mathbb{R}|0 < x < 1\}$ and $\mathbb{R}$ have the same cardinality, $|\mathbb{R}| = |(0,1)|$.

Another example based on $\tan(x)$ and domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $f: (0,1) \rightarrow \mathbb{R}, x \mapsto \tan\left(\frac{x}{\pi} + \frac{\pi}{2}\right)$.

b) For a finite set $A = \{a_1, a_2, \dots, a_N\}$ there is a one-to-one map between subsets and elements of $\{0,1\}^N$, $x = \{x_1, x_2, \dots, x_N\} \in \{0,1\}^N$ corresponds to $\{a_i \in A|x_i = 1\}$.

$$|\mathcal{P}(A)| = |\{0,1\}^N| = 2^N > N = |A|$$

($\{0,1\}^\infty \leftrightarrow$ binary representations of $x \in (0,1) \Longrightarrow |\{0,1\}^\infty| = |(0,1)| = |\mathbb{R}|$)

For an infinite set $|A| < |\mathcal{P}(A)|$ iff there is no surjective function $f: A \rightarrow \mathcal{P}(A)$

$\lvert B\rvert \leq \lvert A\rvert$ iff		$\lvert B\rvert \leq \lvert A\rvert$ iff		$\lvert A\rvert < \lvert B\rvert$ iff
$\exists f: B \rightarrow A$ and	$\Leftrightarrow$	$\exists g: A \rightarrow B$ and	$\Leftrightarrow$	$\neg\exists g: A \rightarrow B$ and
f injecive		g surjective		g surjective

Assume there is a surjective $f: A \rightarrow \mathcal{P}(A)$, $b = \{x \in A|x \notin f(x)\}$ is an element of $\mathcal{P}(A)$.

f surjective $\Rightarrow \exists a \in A: f(a) = b$. $\quad a \in f(a) = b \Rightarrow a \notin f(a)$ contradiction

$a \notin f(a) = b \Rightarrow a \in f(a)$ contradiction

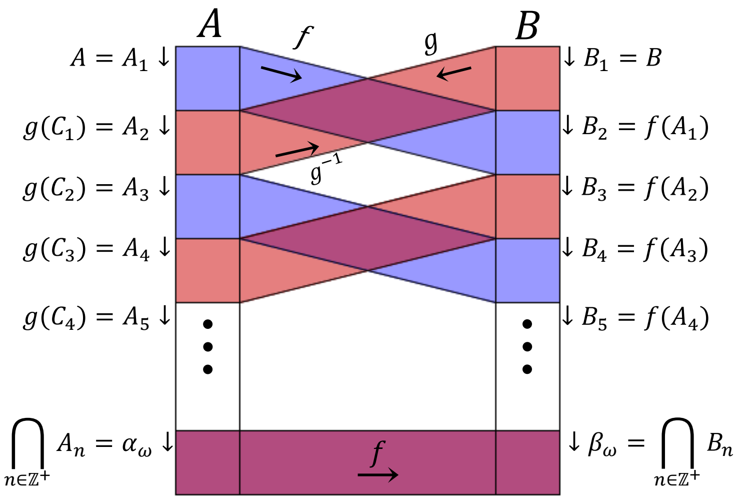
All alternatives leads to contradiction, no surjective $f: A \rightarrow \mathcal{P}(A)$ exists $\Rightarrow |A| < |\mathcal{P}(A)|$.

$|A| < |\mathcal{P}(A)|$ is called Cantor's theorem. The proof used a diagonal argument, like the one used to prove that $\mathbb{R}$ is uncountable by constructing a number outside of any possible listing.

c) $|A| \leq |B|$ and $|B| \leq |A|$ implies that there exists injective functions $f: A \rightarrow B$

$g: B \rightarrow A$

Find a bijective function $h: A \rightarrow B$



Construct partitions of A and B:

$A_1 = A$	$B_1 = B$
$A_{n+1} = g(B_n)$	$B_{n+1} = f(A_n)$
$\alpha_n = A_n \setminus A_{n+1}$	$\beta_n = B_n \setminus B_{n+1}$
$\alpha_\omega = A \setminus \cup_{n=1}^{\infty} \alpha_n$	$\beta_\omega = B \setminus \cup_{n=1}^{\infty} \beta_n$

From their construction we get:

$(\alpha_n)_1^\infty$ and α_ω is a partition of A.

$(\beta_n)_1^\infty$ and β_ω is a partition of B.

$$h_n(x) = \begin{cases} f(x) \text{ if } x \in \alpha_{2k+1} & (k \in \mathbb{N}_0) \\ g^{-1}(x) \text{ if } x \in \alpha_{2k} & (k \in \mathbb{N}_1) \end{cases} \qquad h_\omega(x) = f(x) \text{ if } x \in \alpha_\omega$$

From the construction it follows that $h_n: A \setminus \alpha_\omega \to B - \beta_\omega$ is a bijective function.

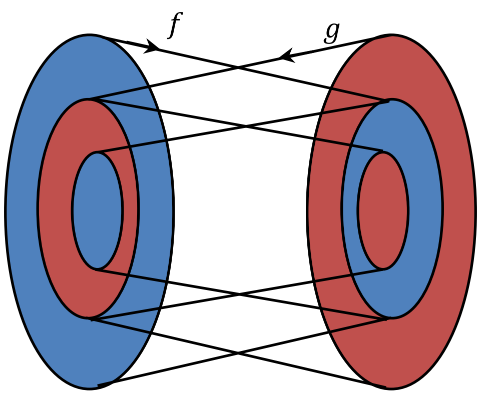
$h_\omega: \alpha_\omega \to \beta_\omega$ is well-defined on α_ω.

$$h_\omega(x) \in \beta_\omega \text{ since } h_\omega(x) \notin \beta_\omega \Rightarrow \left.\begin{matrix} f^{-1}(x) \in A \setminus \alpha_\omega \\ \text{or} \\ g(x) \in A \setminus \alpha_\omega \end{matrix}\right\} \Rightarrow x \notin \alpha_\omega \text{ contradiction, so } h(x) \in \beta_\omega$$

Injectivity of h_ω is inherited from f.

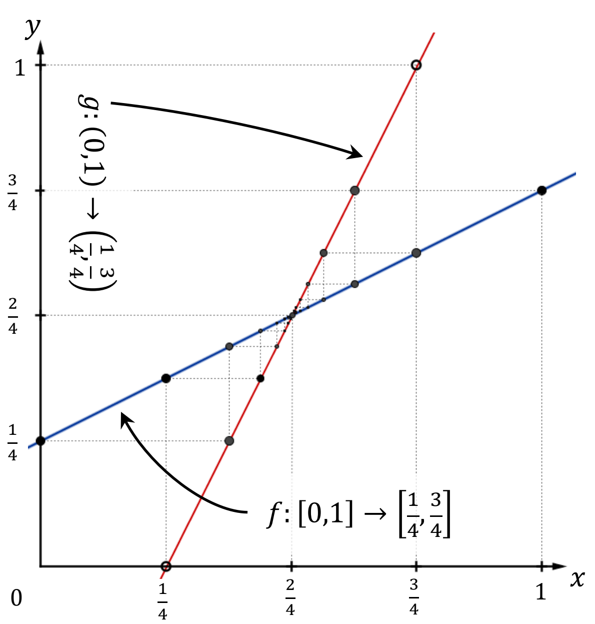
$$\text{Surjectivity: } y \in \beta_\omega, \text{ assume } x = f^{-1}(y) \notin \alpha_\omega \Rightarrow \left.\begin{matrix} f(x) = y \notin \beta_\omega \\ \text{or} \\ g^{-1}(x) = y \notin \beta_\omega \end{matrix}\right\} \text{contradiction, so } f^{-1}(y) \in \alpha_\omega$$

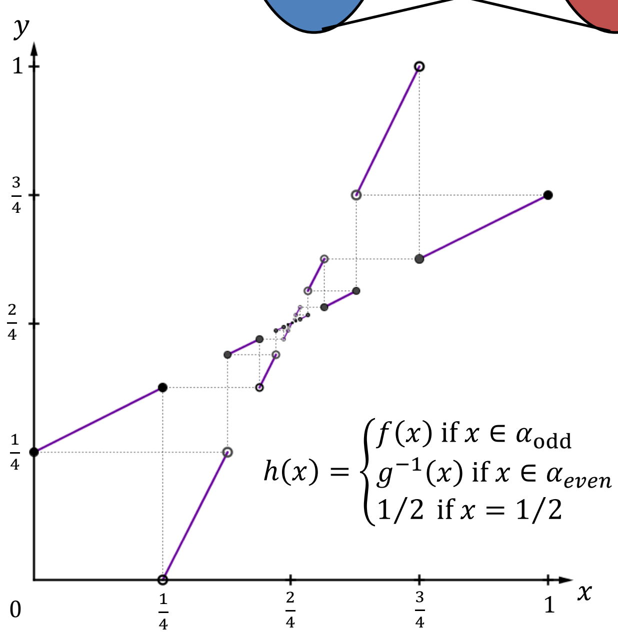
$$h: A \to B, x \mapsto \begin{cases} h_n(x) \text{ if } x \in A \setminus \alpha_\omega \\ h_\omega(x) \text{ if } x \in \alpha_\omega \end{cases} \text{ is a bijection} \Longrightarrow |A| = |B|$$



d)

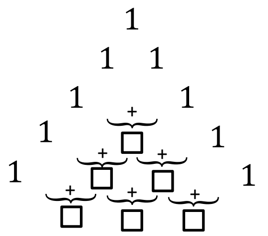
$$f: [0,1] \to (0,1) \qquad g: (0,1) \to [0,1]$$
$$y = x/2 + 1/4 \qquad x = y/2 + 1/4$$
$$f([0,1]) = \left[\frac{1}{4}, \frac{3}{4}\right] \qquad g((0,1)) = \left(\frac{1}{4}, \frac{3}{4}\right)$$
$$f^{-1}: x = 2y - 1/2 \qquad g^{-1}: y = 2x - 1/2$$





3.6 Prove the binomial identities.

1. $\binom{n}{k} = \binom{n}{n-k}$ 2. $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$
3. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ 4. $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$
5. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ 6. $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$
7. $\sum_{m=0}^{n}\binom{m}{r} = \binom{n+1}{r+1}$ 8. $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$



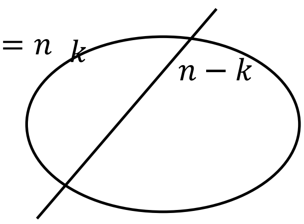
$\rightarrow$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\end{aligned}$$

1. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$

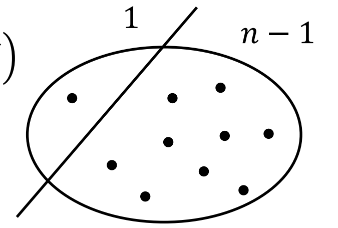
$|Set| = n$



Every subset with k elements corresponds to a subset with $n-k$ elements.
Pascal's triangle is horizontally symmetric.

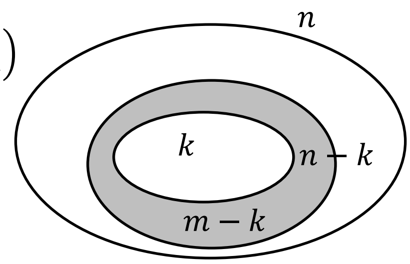
2. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)!}{k\cdot(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$

3. $\underbrace{\binom{n-1}{k-1}}_{A} + \underbrace{\binom{n-1}{k}}_{B} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} = \frac{k(n-1)!+(n-k)(n-1)!}{k!(n-k)!} = \binom{n}{k}$



A is the number of subsets with k elements containing a specified element and B is the number of subsets not containing that element, together they form all subsets of k elements from a set with n elements. This is the defining relation behind Pascal's triangle.

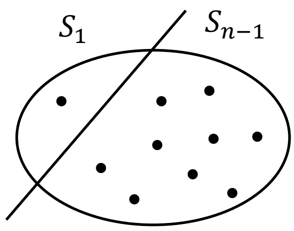
4. $\binom{n}{m}\binom{m}{k} = \frac{n!}{m!(n-m)!}\cdot\frac{m!}{k!(m-k)!} = \frac{n!}{k!(n-k)!}\cdot\frac{(n-k)!}{(n-m)!(m-k)!} = \binom{n}{k}\binom{n-k}{m-k}$



$\binom{n}{m}\binom{m}{k}$ ways to first pick a subset with m elements and then choose a subset of that subset with k elements. Each chosen set will be double-counted by a factor $\binom{n-k}{m-k} \rightarrow \binom{n}{k} = \binom{n}{m}\binom{m}{k} / \binom{n-k}{m-k}$

5. $2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}$ Sum of elements in a row of Pascal's triangle
The sum counts all possible subsets of a set of n elements, i.e. 2^n.

6. $0 = (1-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k1^{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$



$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0 \leftrightarrow \sum_{k \text{ is even}}\binom{n}{k} = \sum_{k \text{ is odd}}\binom{n}{k}$

$\#\text{even subsets of } S_n = \underbrace{\#\text{odd subsets of } S_{n-1}}_{S_1 \text{ object included}} + \underbrace{\#\text{even subsets of } S_{n-1}}_{S_1 \text{ object excluded}}$

$\#\text{odd subsets of } S_n = \#\text{even subsets of } S_{n-1} + \#\text{odd subsets of } \mathrm{S}_{n-1}$

7. Proof by induction

$n = 0$: $\binom{0}{r} = \binom{1}{r+1}$

Both sides equals δ_{r0}.

If statement is true for $n-1$,

$$\sum_{m=0}^{n-1}\binom{m}{r} = \binom{n}{r+1}$$

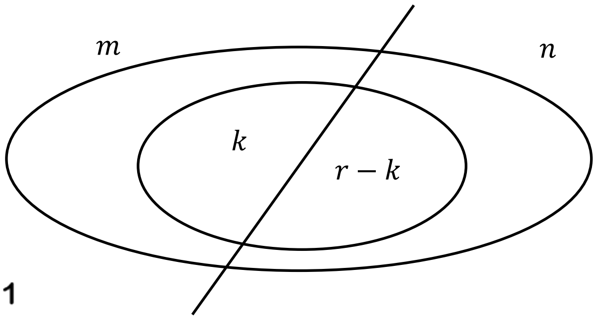
$$\sum_{m=0}^{n-1}\binom{m}{r} + \binom{n}{r} = \binom{n}{r+1} + \binom{n}{r}$$

$$\sum_{m=0}^{n}\binom{m}{r} = \binom{n+1}{r+1} \quad \text{then it is true for } n.$$

8. Show $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$

Divide S_{m+n} in two parts, A and B.

All subsets of S_{m+n} with r elements have k elements in A, and $r-k$ elements in B for some $0 \le k \le r$. Sum them up



$\binom{0}{0}$

$\binom{1}{0}$ $\binom{1}{1}$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Pascal's triangle with binomial coefficients n-choose-k

3.7 Prove the multinomial theorem:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$$

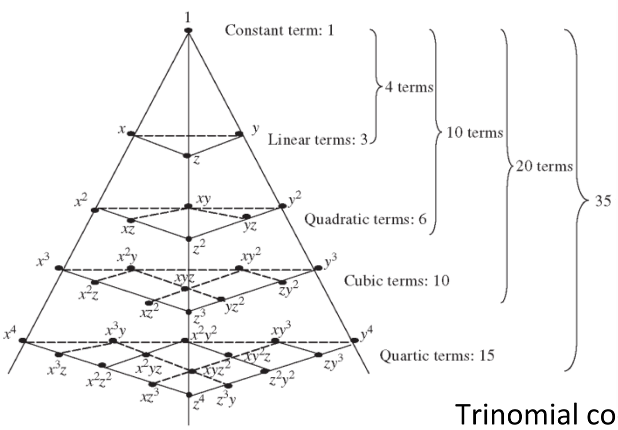
$(x_1 + x_2 + \cdots + x_m) \cdot \ldots \cdot (x_1 + x_2 + \cdots + x_m)$, n factors

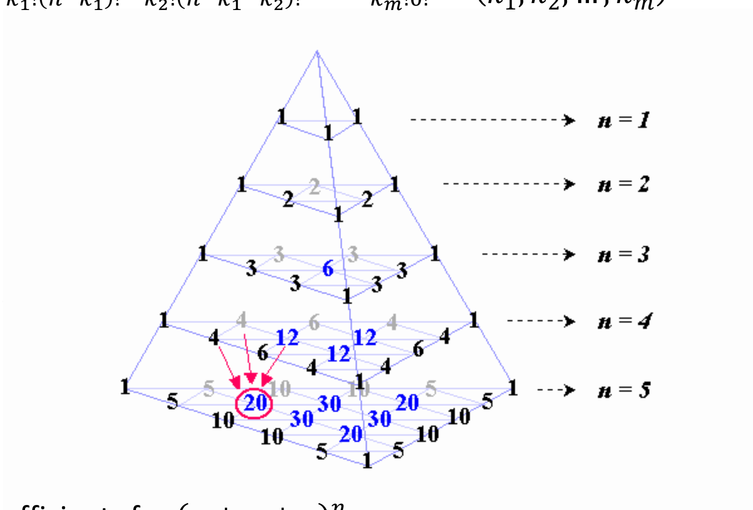
Using the distributive property of numbers gives a sum of terms $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$.

Each parentheses contributes one x_i so $k_1 + k_2 + \cdots + k_m = n$.

Pick x_1 from k_1 parentheses out of n, then pick k_2 from remaining $n - k_1$, ….

$$\binom{n}{k_1}\cdot\binom{n-k_1}{k_2}\cdot\binom{n-k_1-k_2}{k_2}\cdot\ldots\cdot\binom{k_m}{k_m} = \frac{n!}{k_1!(n-k_1)!}\cdot\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}\cdot\ldots\cdot\frac{k_m!}{k_m!0!} = \binom{n}{k_1, k_2, \ldots, k_m}$$



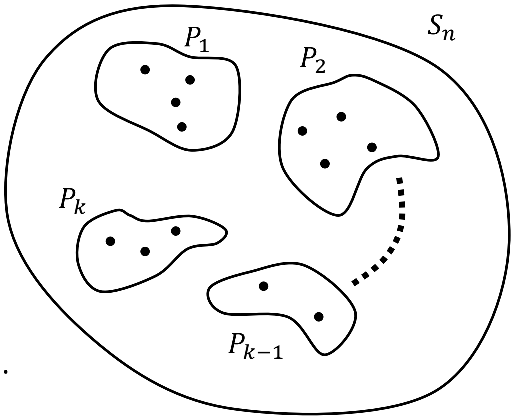


Trinomial coefficients for $(x+y+z)^n$

3.8 Stirling numbers $\left\{{n \atop k}\right\}$ of the second kind are defined as the number of ways to partition a set of n objects $S_n = \{1,2,\dots,n\}$ into k non-empty subsets. Show that k!$\left\{{n \atop k}\right\}$ equals the number of surjective functions $f: S_n \to S_k$ and that

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

Each surjective function $f: S_n \to S_k$ belongs to a partition $S_n = \bigcup_{i=1}^{k} P_i$ and $f(P_i) = \{j\}$ where each $j \in S_k$ is in the image of some partition set. The number of partitions is $\left\{{n \atop k}\right\}$, by definition. There are $k!$ possible surjective functions for each partition (by permuting the values $j \in S_k$).



∴ There are $N = k!\left\{{n \atop k}\right\}$ surjective functions $f: S_n \to S_k$.

Let's count them in another way. A combinatorial problem of counting that leads to a sum with terms of alternating signs suggests that the inclusion-exclusion principle might be involved. Let $\{T_j | 1 \le j \le n\}$ be a collection of sets. The number of elements in their union is:

$$\left|\bigcup_{j=1}^{n} T_j\right| = \sum_{k=1}^{n}\left[(-1)^{k+1}\left(\sum_{1 \le j_1 < \cdots < j_k \le n}\left|S_{j_1} \cap \dots \cap S_{j_k}\right|\right)\right]$$

Let X be the set of functions $f: S_n \to S_k$, $|X| = k^n$ and
let $X_j = \{f: S_n \to S_k | f(S_n) \cap \{j\} = \emptyset\}$ (j is not in the image set of f)
The number of surjective functions $f: S_n \to S_k$ is:

$$N = \left|X \setminus \bigcup_{j=1}^{k} X_j\right| = k^n - \left|\bigcup_{j=1}^{n} X_j\right|$$

$|X_j| = (k-1)^n$ and by extending this to several excluded elements gives:

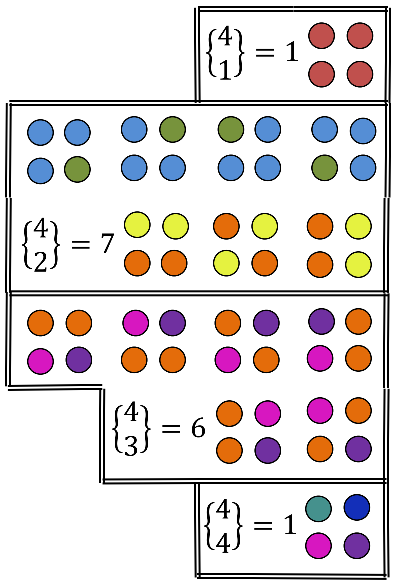
$$\sum_{1 \le i_1 \le \dots \le i_j \le k}\left|X_{i_1} \cap X_{i_2} \cap \dots \cap X_{i_k}\right| = \binom{k}{j}(k-j)^n$$

By the inclusion-exclusion principle we get:

$$N = k^n - \sum_{j=1}^{k}(-1)^{j+1}(k-j)^n = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$

Finally, equating the two counts gives:

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$$



3.9

According to legend there is a temple with monks and 64 golden disks resting on three pillars. Ancient rules dictate that a disk may never rest on a smaller disk. When all disks have been moved the world will end. They are working day and night moving one disk every second. What is the shortest time to move all 64 golden disks?

Let a_n be the minimal number of moves to move n disks from one pillar to another. To do this $n-1$ disks must first be moved to another pillar so that the bottom disk can be moved then the $n-1$ disks must be relocated to the pillar with the big disk.

$a_n = 2a_{n-1} + 1$
$a_1 = 1$

Method 1
Find the general solution to the homogenous equation and use an ansatz to find a particular solution to the inhomogeneous equation.
$a_n - 2a_{n-1} = 0 \to a_n = C \cdot 2^n$
Ansatz for particular solution $a_n = x \to x = 2x + 1 \to x = -1 \to a_n = C \cdot 2^n - 1$
$a_1 = 1 \to C = 1 \to a_n = 2^n - 1$
Method 2
Rewrite the recurrence relation to get a homogeneous equation and solve it by methods familiar from differential equations (characteristic equation).

$$\begin{matrix} a_n = 2a_{n-1} + 1 \\ a_{n+1} = 2a_n + 1 \end{matrix} \to a_{n+1} - a_n = 2a_n - 2a_{n-1} \to \begin{matrix} a_{n+1} - 3a_n + 2a_{n-1} = 0 \\ r^2 - 3r + 2 = 0 \to \end{matrix} \quad r_1 = 1\,, r_2 = 2$$

$$a_n = C_1 1^n + C_2 2^n \;\; \begin{matrix} a_1 = 1 \\ a_3 = 3 \end{matrix} \to \begin{matrix} C_1 = -1 \\ C_2 = 1 \end{matrix} \to a_n = 2^n - 1$$

Method 3
Use a generating function $G(z)$ for $\langle a_n \rangle$

$$\langle a_n \rangle = 2\langle a_{n-1} \rangle + \langle 0,1,1,\dots \rangle \quad (a_{-1} = 0, a_0 = 0, a_1 = 1) \to \quad G(z) = 2zG(z) + \frac{z}{1-z} \to$$

$$G(z) = \frac{z}{(1-z)(1-2z)} = \frac{A}{1-z} + \frac{B}{1-2z} \to \begin{matrix} A = -1 \\ B = 1 \end{matrix} \to G(z) = \frac{-1}{1-z} + \frac{1}{1-2z} \to$$

$$G(z) = -\sum\nolimits_k z^k + \sum\nolimits_k (2z)^k \to G(z) = \sum\nolimits_k (2^k - 1)z^k \to a_n = 2^n - 1$$

The time in years to move 64 disks from one pillar to another is $(2^{64} - 1)/(60 \cdot 60 \cdot 24 \cdot 365.25)$. This is $5.8 \cdot 10^{11}$ years or 42 times the current age of the universe. The *Towers of Hanoi* puzzle with 8 disks was invented or at least popularized by French mathematician Edouard Lucas in 1883. He also added the mythical story *Tower of Brahma* with 64 disks and monks or priests constantly moving disks from the beginning of time to the end of time.

3.10 How many different messages of length n can be built from two symbols of of length 1 and length 2?

$n = 1$, {  }

$n = 2$, {  ,  }

Message of length 12

Compare the growth rate with a geometric sequence.

Let F_n be the number of messages of length n.
Every message of length n ends either with a short signal with F_{n-1} possible earlier combinations or a long signal with F_{n-2} possible preceding combinations.

$F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. $\langle F_n \rangle = \langle 0,1,1,2,3,5,8,13,21,34, \dots \rangle$

$$F_n - F_{n-1} - F_{n-2} = 0$$
$$r^2 - r - 1 = 0$$
$$r = \frac{1 \pm \sqrt{5}}{2}$$
$$F_n = C_1 r_1^n + c_2 r_2^n$$

$$\begin{matrix} F_1 = 0 \\ F_2 = 0 \end{matrix} \rightarrow \begin{matrix} C_1 = 1/\sqrt{5} \\ C_2 = -1/\sqrt{5} \end{matrix}$$

$$F_n = 2^{-n}\sqrt{5}\left(\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n\right)$$

The sequence known as the Fibonacci sequence was studied by Indian mathematicians long before Fibonacci introduced it to Europe in 1202 with an example of rabbit reproduction.

"Towers of Hanoi"-Lucas from the previous problem studied them thoroughly. It was he who popularized the term "Fibonacci numbers". He used them to prove that the 39-digit number $2^{127} - 1$ is a prime number.

Pingala, Indian scholar of mathematics and Sanskrit came across Fibonacci numbers already in the 2nd century BC when he studied poetic metres with short and long syllables. Other achievements by Pingala are recursion, a binary numeral system and the binomial theorem.

If the ratio of adjacent numbers F_{n+1}/F_n has a limit x, it should satisfy:

$$\frac{F_{n+1}}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n} \rightarrow x = 1 + \frac{1}{x} \text{ as } n \rightarrow \infty \quad \Rightarrow x = \frac{1+\sqrt{5}}{2}$$

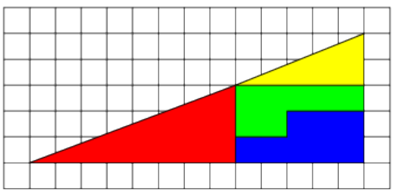
$$\lim_{n\to\infty} F_{n+1}/F_n = \lim_{n\to\infty} \frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}} \frac{2^n}{\left(1+\sqrt{5}\right)^n} = \frac{1+\sqrt{5}}{2}$$

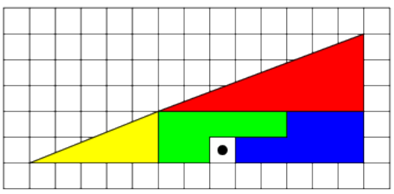
The Fibonacci number growth approach a geometric series with ratio equal to the golden ratio $\varphi = \left(1+\sqrt{5}\right)/2$

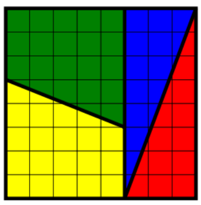
One of many formulas with Fibonacci numbers is Cassini's identity:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

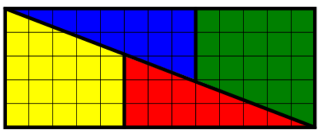
It can be used to construct the following rearrangement puzzles:







8^2 rutor



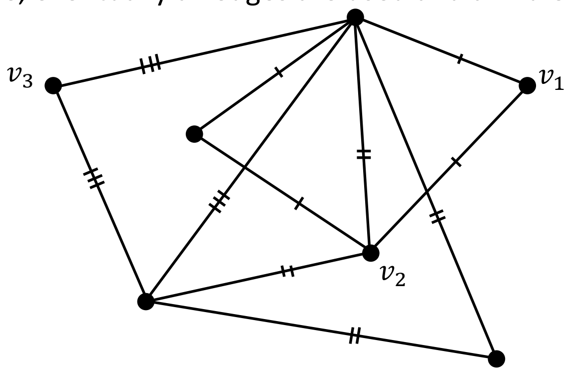
$5 \cdot 13$ rutor

3.11 Prove Euler-Hierholzer's theorem from graph theory. A connected graph $G = (V, E)$ has an Euler cycle if and only if every vertex is of even degree.

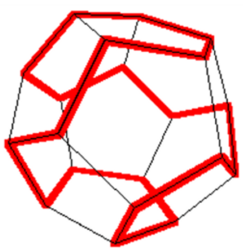
An Euler cycle is a path that starts and ends in the same vertex and contains every edge once.

$\Longrightarrow$ (Assume G has an Euler cycle γ and show that every vertex is of even degree)
Choose a direction for γ. Every vertex v belongs to γ and all edges incident with v are either ingoing or outgoing, these sets are disjoint and of the same size so $\deg(v)$ must be even.

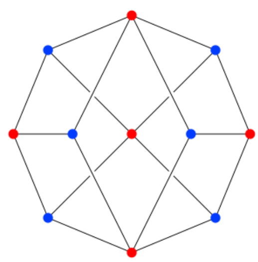
$\Longleftarrow$ (Assume every vertex of G is of even degree and show that there is an Euler cycle in G)
Pick a vertex v_1 and add edges incrementally until there are no edges to continue with.
All vertices are of even degree $\Longrightarrow$ the path will end in v_1, it is a cycle γ_1.
If all edges in E used, we are done.
If not, form a subgraph H with V and the unused edges of E. All vertices in H have even degree.
If $\gamma_1$had no edges connecting it to H it would be a disconnected piece of G which is contradictory.
Choose a vertex v_2 in $\gamma_1$with an edge in H and form a cycle γ_2 as before, starting with this edge.
γ_1 and γ_2 can be combined into one cycle γ.
If all edges are used we are done.
If not we can form a new subgraph of unused edges and form a new cycle γ_3 and absorb it into γ.
The number of edges is finite, eventually all edges are used and an Euler cycle is formed. ■



The corresponding question for a Hamitonian cycle has no known non-trivial condition that is both necessary and sufficient. A Hamitonian cycle starts and ends in the same vertex and passes each vertex exactly once. The name comes from William Hamilton who invented the icosian game where the vertices in a dodecahedron represent 20 cities and the goal of the games is to make a round trip along the dodecahedron edges and visit each city exactly once. (20=icosa in Greek).



To the left a Hamiltonian cycle on a dodecahedron and to the right a Herschel graph which is the smallest non-Hamiltonian polyhedral graph. A polyhedral graph is based on vertices and edges of a convex polyhedron.



If you can remove k vertices and their incident edges and the graph become disconnected into $k + 1$ or more connected pieces then there can be no Hamiltonian cycle. (necessary condition)
If G is a loop-free graph with $n \geq 3$ vertices and for every pair x, y of non-connected vertices $\deg(x) + \deg(y) \geq n$ then there will be a Hamiltonian cycle. (sufficient condition)

3.12 Show that the set of numbers $\mathbb{Q}[\sqrt{2}] := \{q_1 + q_2\sqrt{2} | q_1, q_2 \in \mathbb{Q}\}$ form a field under ordinary addition and multiplication.

One way is to do a direct check of all the field axioms. Another way is to establish a link to matrices and rely on their properties under matrix addition $\oplus$ and matrix multiplication $\odot$, in the same way that complex numbers $z = a + bi \in \mathbb{R}[i]$ corresponds to $z_M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Let $q = q_1 + q_2\sqrt{2}$ correspond to $q_M = \begin{pmatrix} q_1 & 2q_2 \\ q_2 & q_1 \end{pmatrix} \in M_{\sqrt{2}} := \left\{ \begin{pmatrix} \alpha & 2\beta \\ \beta & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{Q} \right\}$

$q + r = (q_1 + r_1) + (q_2 + r_2)\sqrt{2} \in M_{\sqrt{2}}$ (closure under addition)

$q \cdot r = (q_1 r_1 + 2q_2 r_2) + (q_1 r_2 + q_2 r_1)\sqrt{2} \in M_{\sqrt{2}}$ (closure under multiplication)

$$q_M \oplus r_M = \begin{pmatrix} q_1 + r_1 & 2(q_2 + r_2) \\ q_2 + r_2 & q_1 + r_1 \end{pmatrix} = (q + r)_M$$

$$q_M \odot r_M = \begin{pmatrix} q_1 & 2q_2 \\ q_2 & q_1 \end{pmatrix} \odot \begin{pmatrix} r_1 & 2r_2 \\ r_2 & r_1 \end{pmatrix} = \begin{pmatrix} q_1 r_1 + 2q_2 r_2 & 2(q_1 r_2 + q_2 r_1) \\ q_2 r_1 + q_1 r_2 & 2q_2 r_2 + q_1 r_1 \end{pmatrix} = (q \cdot r)_M$$

Associative, commutative and distributive properties for addition and multiplication in $\mathbb{Q}[\sqrt{2}]$ follows from the corresponding properties of the corresponding matrix operators.
(Matrix multiplication is generally not commutative but those in the subset $M_{\sqrt{2}}$ are.)

The additive identity is $0 + 0\sqrt{2} \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{\sqrt{2}}$

The multiplicative identity is $1 + 0\sqrt{2} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{\sqrt{2}}$

The additive inverse of $q_1 + q_2\sqrt{2}$ is $-q_1 + (-q_2)\sqrt{2} \sim \begin{pmatrix} -q_1 & -2q_2 \\ -q_2 & -q_1 \end{pmatrix} \in M_{\sqrt{2}}$

The multiplicative inverse of $q_1 + q_2\sqrt{2}$ with $q_1, q_2 \neq 0$ is $(q_1^2 - 2q_2^2)^{-1}(q_1 - q_2\sqrt{2}) \in M_{\sqrt{2}}$

$\begin{vmatrix} q_1 & 2q_2 \\ q_2 & q_1 \end{vmatrix} = q_1^2 - 2q_2^2 \qquad q_1^2 - 2q_2^2 = 0 \Rightarrow q_1/q_2 = \sqrt{2}$ or $q_1 = q_2 = 0$ which is contradictory

This could of course have been done much more easily by embedding $\mathbb{Q}[\sqrt{2}]$ in $\mathbb{R}$ and showing that $q + r, q \cdot r, 0, -q, 1, q^{-1}$ belongs to $\mathbb{Q}[\sqrt{2}]$ and appealing to the properties of commutativity, associativity and distributivity under addition and multiplication in $\mathbb{R}$ but the method is useful in the search for extensions of $\mathbb{C}$.

W.R. Hamilton, inventor of the icosian game extended the complex numbers into a larger but non-commutative field, the quaternion number system $\mathbb{H} := \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} | a, b, c, d \in \mathbb{R}\}$.
Addition is defined component-wise. $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \Longrightarrow$
Multiplication is defined by the distributivity and multiplication table →

×	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1

$\mathbb{H}$ can be embedded into $M(2, \mathbb{C})$ the complex 2×2 matrices via
$\begin{pmatrix} z & w \\ \underline{w} & \overline{z} \end{pmatrix}$, $z = a + bi$, $w = c + di$, $\overline{z} = a - bi$, $\underline{w} = -c + di$.

$$\begin{pmatrix} z & w \\ \underline{w} & \overline{z} \end{pmatrix}^{-1} = \frac{1}{|z|^2 + |w|^2} \begin{pmatrix} \overline{z} & -w \\ \overline{w} & z \end{pmatrix}$$

$$\begin{aligned} q &= a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \\ \overline{q} &= a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k} \quad \rightarrow q^{-1} = \overline{q}/|q|^2 \\ |q|^2 &= a^2 + b^2 + c^2 + d^2 \end{aligned}$$

3.13 Show equivalence of the different definitions of multiplicity k for roots of $P(z)$.

$$\begin{matrix} (z-\alpha)^k \mid P(z) \\ (z-\alpha)^{k+1} \nmid P(z) \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} P^{(i)}(\alpha) = 0 \ \text{ for } i \in \{0,1,\dots,k-1\} \\ P^{(k)}(\alpha) \neq 0 \end{matrix}$$

$\Longrightarrow$

$P(z) = (z-\alpha)^k Q_0(z)$ with $Q_0(\alpha) \neq 0$ $\qquad \Rightarrow P^{(0)}(\alpha) = 0$

$P'(z) = (z-\alpha)^{k-1}\underbrace{(kQ_0(z) + (z-\alpha)Q_0'(z))}_{Q_1(z)}$ $\qquad \Rightarrow P^{(1)}(\alpha) = 0$

$P''(z) = (z-\alpha)^{k-2}\underbrace{((k-1)Q_1(z) + (z-\alpha)Q_1'(z))}_{Q_2(z)}$ $\qquad \Rightarrow P^{(2)}(\alpha) = 0$

• • •

$P^{(k-1)}(z) = (z-\alpha)\underbrace{(2Q_{k-2}(z) + (z-\alpha)Q_{k-2}'(z))}_{Q_{k-1}(z)}$ $\qquad \Rightarrow P^{(k-1)}(\alpha) = 0$

$P^{(k)}(z) = \underbrace{1 \cdot Q_{k-1}(z) + (z-\alpha)Q_{k-1}'(z)}_{Q_k(z)}$ $\qquad \Rightarrow P^{(k)}(\alpha) = Q_{k-1}(\alpha) = 2Q_{k-2}(\alpha) =$

$2 \cdot 3 \cdot Q_{k-3}(\alpha) = \cdots = 2 \cdot 3 \ldots \cdot k \cdot Q_0(\alpha) \neq 0$

$\Longleftarrow$

$P(\alpha) = 0 \Rightarrow$ $\qquad P(z) = (z-\alpha)R_1(z)$

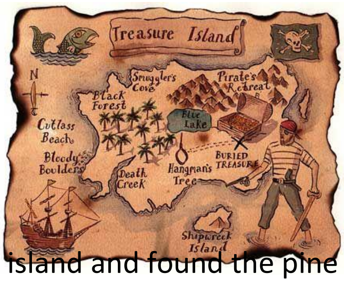
$P'(\alpha) = 0 \Rightarrow$ $\qquad R_1(\alpha) = 0 \Rightarrow$ $\qquad P(z) = (z-\alpha)^2 R_2(z)$

• • •

$P^{(k-1)}(\alpha) = 0 \Rightarrow$ $\qquad R_{k-1}(\alpha) = 0 \Rightarrow$ $\qquad P(z) = (z-\alpha)^k R_k(z)$

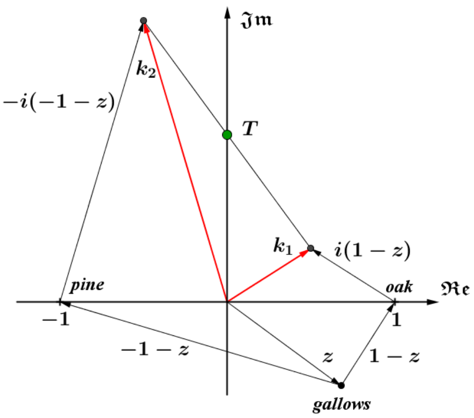
$P^{(k)}(\alpha) \neq 0 \Rightarrow$ $\qquad R_k(\alpha) \neq 0 \Rightarrow$ $\qquad (z-\alpha)^{k+1} \nmid P(z)$

3.14 The location of a pirate treasure is described as follows:
Go from the gallows to the oak, turn 90 degrees to the left, walk the same distance and put a knife in the ground. Go back to the gallows, walk to the pine, turn 90 degrees to the right, walk the same distance and put another knife in the ground. Midway between the knives, dig and you will find the treasure.



Descendants of the pirate found the description. They went to the island and found the pine and the oak but no gallows but still they could find the treasure. Describe where they found it.

Introduce a complex plane and locate the real axis so that the oak is at 1 and the pine is at -1.



Rotate a vector by multiplying with $e^{i\varphi}$.
The location of various objects will be as follows.
Gallows: z
First knife: $k_1 = 1 + i(1-z)$
Second knife: $k_2 = -1 + (-i)(-1-z)$
Treasure: $T = \frac{1}{2}(k_1 + k_2) = i$

Go from the oak towards the pine. When you are half-way, turn right 90°, walk the same distance and start to dig.

3.15 Show $e^z e^w = e^{z+w}$ for $z, w \in \mathbb{C}$.

---

e^z is defined by extending $e^x = \sum_{k=0}^{\infty} x^k/k!$ from $x \in \mathbb{R}$ to $z \in \mathbb{C}$.
A series such as $\sum_{k=0}^{\infty} z^k/k!$ is said to be convergent if the partial sums $S_n = \sum_{k=0}^{n} z^k/k!$ tend to a limit $S = \lim\limits_{n\to\infty} S_n$ which means that $\forall \varepsilon \in \mathbb{R}^+ \exists n \in \mathbb{N}: n > N \Rightarrow |S_n - S| < \varepsilon$.
An infinite series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ is convergent.
The terms can then be summed in any order without affecting convergence or the limit.
The ratio test for successive terms in the exponential series gives $\frac{|a_{n+1}|}{|a_n|} = \frac{x}{n+1} \to 0$ as $n \to \infty$ so the series is absolutely convergent for all $z \in \mathbb{C}$, and both sides of the relation are well-defined.

$$e^{z+w} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!\,(n-k)!} =$$

Setting $j = n - k$ and reshuffling the summation order gives:

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{z^k w^j}{k!\,j!} = \sum_{j=0}^{\infty} \left(\frac{w^j}{j!} \sum_{k=0}^{\infty} \frac{z^k}{k!} \right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \sum_{j=0}^{\infty} \frac{w^j}{j!} = e^z e^w$$

$e^z e^{-z} = e^0 = 1 \to e^z \neq 0$ and $1/e^z = e^{-z}$
Continuity of the conjugation operation gives $e^{\bar{z}} = \overline{e^z}$
For $t \in \mathbb{R}$, the conjugate of e^{it} is e^{-it}, hence $|e^{it}| = 1$.
e^{it} lies on the unit circle which suggest the following:

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}$$

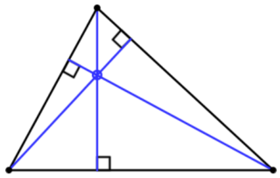
$\cos z = \Re(e^{iz})$ and $\sin z = \Im(e^{iz})$ for any $z \in \mathbb{C}$.
With trigonometric functions defined in terms of e^z all trigonometric identities become consequences of

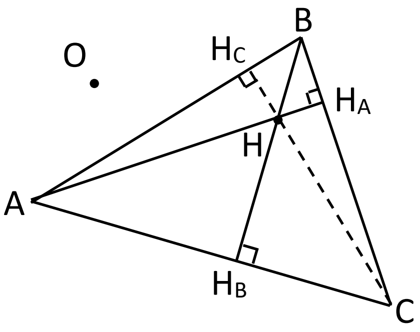
$$e^z e^w = e^{z+w} \text{ and } e^{\bar{z}} = \overline{e^z}$$

3.16 A Graeco-Latin square or an Euler square of order n is an arrangement of symbols from $\mathrm{G}=\{\alpha,\beta,\gamma,\dots\}$ and $\mathrm{L}=\{a,b,c,\dots\}$ with $|\mathrm{G}|$=$|\mathrm{L}|$=n in such a way that each cell of an $n\times n$ square contains an ordered pair $(g,l)\in\mathrm{G}\times\mathrm{L}$. Every row and every column contain each element of G and each element of L exactly once and no cells contain the same pair. Euler presented the problem for $n=6$ with $\mathrm{G}=\{\text{officer ranks}\}$ and $\mathrm{L}=\{\text{regiments}\}$, "the thirty-six officers' problem". He constructed Graeco-Latin squares for n=$2k+1$ and n=$4k$ Euler conjectured that no Graeco-Latin squares exists for n=$4k+2$. Show that he was wrong!

A similar problem with n=4 and 16 playing cards, $\mathrm{G}=\{\mathrm{A},\mathrm{K},\mathrm{Q},\mathrm{J}\}$ and $\mathrm{L}=\{\clubsuit,\diamondsuit,\heartsuit,\spadesuit\}$ has an extra constraint; each diagonal should also contain all four face values and all four suits. How many solutions are there?

3.17 Show that the three altitudes of a triangle have one point in common, (the orthocenter).

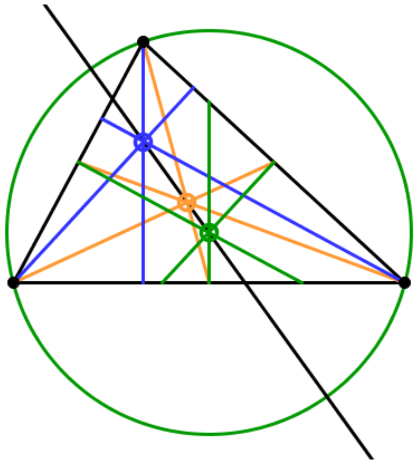


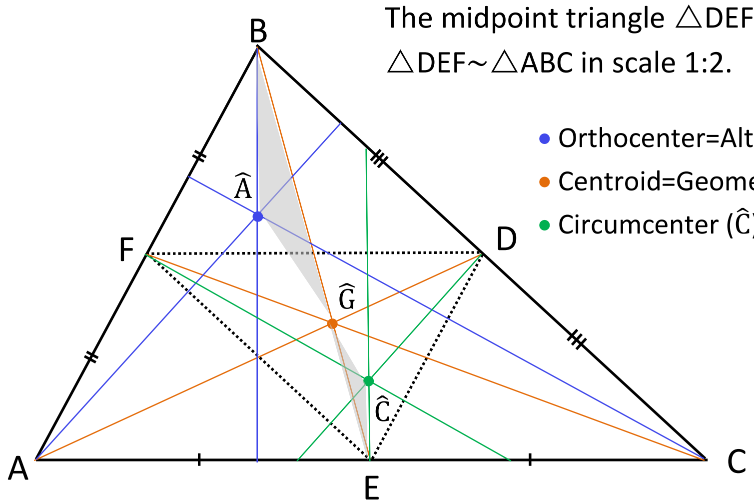


Select a point O as origin.
Let H be the intersection of AH_A and BH_B.
(1) $AH \perp BC \Rightarrow (OH-OA)\cdot(OC-OB)=0$
(2) $BH \perp AC \Rightarrow (OH-OB)\cdot(OC-OA)=0$

Subtract equation (1) from equation (2) and expand.
$OH\cdot OB+OA\cdot OC-OH\cdot OA-OB\cdot OC=0$
$OH\cdot(OB-OA)+OC\cdot(OA-OB)=0$
$(OH-OC)\cdot(OB-OA)=0 \Rightarrow CH\perp AB \Rightarrow$ H lies on CH_C

3.18 Show that orthocenter, centroid and circumcenter of a non-equilateral triangle are collinear.





The midpoint triangle $\triangle$DEF is similar to triangle $\triangle$ABC.
$\triangle DEF \sim \triangle ABC$ in scale 1:2.

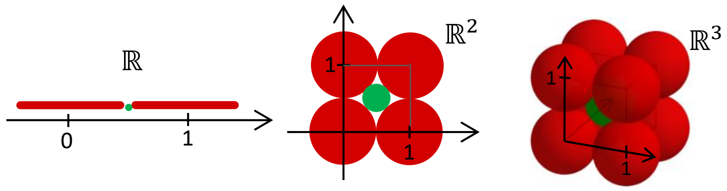
- Orthocenter=Altitude center ($\hat{A}$)
- Centroid=Geometric center ($\hat{G}$)
- Circumcenter ($\hat{C}$) of $\triangle$ABC = Altitude center of $\triangle$DEF

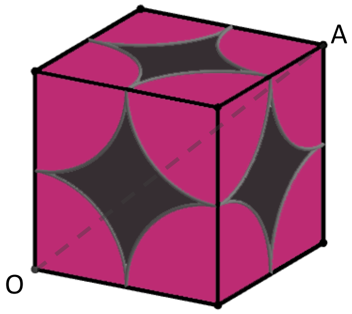
Proof of $\triangle B\hat{A}\hat{G} \sim \triangle E\hat{C}\hat{G}$:
$\hat{A}B \parallel E\hat{C} \Rightarrow \angle \hat{A}B\hat{G} = \angle \hat{C}E\hat{G}$
Geometric center has $\|B\hat{G}\| = 2\|\hat{G}E\|$
$B\hat{A}$ and $E\hat{C}$ are corresponding segments from vertex to altitude center in similar triangles of ratio 2:1 so $\|B\hat{A}\| = 2\|E\hat{C}\|$.

$\therefore \triangle B\hat{A}\hat{G} \sim \triangle E\hat{C}\hat{G}$

$\triangle B\hat{A}\hat{G} \sim \triangle E\hat{C}\hat{G} \Rightarrow \angle \hat{A}\hat{G}B = \angle \hat{C}\hat{G}E \Rightarrow \hat{A}, \hat{G}$ and $\hat{C}$ are collinear.
The line that they sit on is called **Euler's line**.

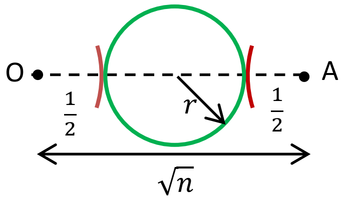
3.19. Explore how the radius varies with dimension for a sphere that is squeezed in between spheres centered at integer coordinates $\mathbb{Z}^n$ in $\mathbb{R}^n$.





The sphere at the center of the n-cube is touches the spheres centered at the corners of the unit n-cube.

Length of space diagonal in n dimensions: $\underbrace{\sqrt{1^2+\cdots+1^2}}_{n\text{ terms}} = \sqrt{n}$



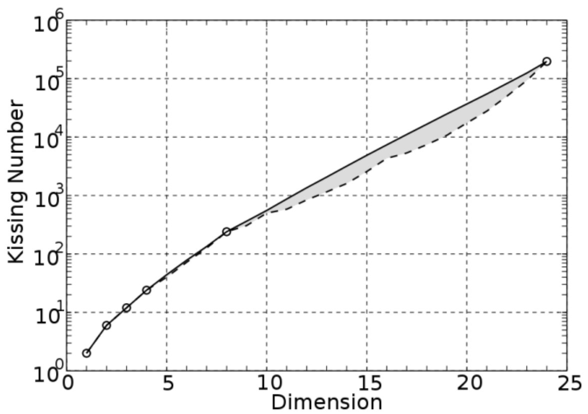
The radius of the inner sphere is:

$r = \frac{\sqrt{n}-1}{2} \to \infty$ as $n \to \infty$

Strictly increasing while spheres at the vertices are fixed with $r = ½$.

As the dimension increases the inner sphere will grow without limit in the axis directions but its center will always stay half a unit from the vertices of the n-cube.
In 4 dimensions $n = 4$ and $r = ½$. The inner sphere will reach the faces of the 4-cube and it will be tangential to the 16 spheres at the vertices and 8 spheres in the x,y,z,w-directions.
This arrangement is the answer in four dimensions to the kissing number problem:
How many non-overlapping unit spheres can be arranged so that they each touch another unit sphere?
The answer is only known for a few dimensions, for other dimensions there is upper and lower bounds.

Dimension	Lower bound	Upper bound
1	2	
2	6	
3	12	
4	24	
5	40	44
6	72	78
7	126	134
8	240	
24	196,560	



Kissing number growth, probably exponential.

The known solutions for 8 and 24 dimensions correspond to the E_8 lattice and the Leech lattice.
The 4-dimensional regular arrangement is also the answer to the problem of finding the densest possible arrangement of 4-spheres in 4 dimensions, the hypersphere packing problem.
The densest regular sphere packing problem is solved for dimensions 1 to 8 and 24.
With irregular packings included the answer is only proved for dimensions 1, 2, 3, 4, 8 and 24.

3.20. $f: X \to Y$ is a function between two metric spaces with $\|a-b\| = d(a,b)$

Show that the following definitions of $\lim_{x \to x_0} f(x) = y_0$ are equivalent.

A. For every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < \|x - x_0\|_X < \delta \Longrightarrow \|f(x) - y_0\|_Y < \epsilon$

B. For every neighborhood $\mathcal{V}$ of $y_0$there is a punctured neighborhood $\mathcal{U}$ of x_0 s.t. $f(\mathcal{U}) \subseteq \mathcal{V}$

nh ≡ neighborhood	pnh ≡ punctured neighborhood	os ≡ open set	ip ≡ interior point

Assume A

$\mathcal{V}$ nh of $y_0 \Rightarrow \exists$ os $\mathcal{V}_0 \subseteq \mathcal{V}$ with $y_0 \in \mathcal{V}_0 \;\Rightarrow y_0$ ip of $\mathcal{V}_0 \;\Rightarrow\; \exists \epsilon' > 0 : \mathrm{B}_{\epsilon'}(y_0) \subseteq \mathcal{V}_0 \;\Longrightarrow$ (By A)

$\exists \delta' > 0 : \underbrace{(0 \leq \|x - x_0\|_X < \delta' \Rightarrow \|f(x) - y_0\|_Y < \epsilon')}_{\text{pnh of } x_0\ (=\mathcal{U}) \text{ mapped into } \mathrm{B}_{\epsilon'}(y_0) \subseteq \mathcal{V}_0 \subseteq \mathcal{V}} \Longrightarrow \exists$ pnh $\mathcal{U}$ of x_0 s.t $f(\mathcal{U}) \subseteq \mathcal{V}$

so A⇒B

Assume B

$\epsilon > 0 \Rightarrow \mathrm{B}_\epsilon(y_0)$ nh of $y_0 \;\Longrightarrow$ (By B) $\exists$ pnh $\mathcal{U}$ of x_0 s.t. $f(\mathcal{U}) \subseteq \mathrm{B}_\epsilon(y_0) \;\Longrightarrow$

[N.B. If x_0 is added to $\mathcal{U}$ there is an open set containing x_0 inside $\mathcal{U}$ i.e. x_0 is an interior point, it has a punctured open ball around x_0 mapped into $\mathrm{B}_\epsilon(y_0)$]

$\exists \delta > 0 :\; (0 < \|x - x_0\|_X < \delta \Longrightarrow \|f(x) - y_0\|_Y < \epsilon)$

so B⇒A A⇒B and B⇒A so A⇔B

3.21 Show that if $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$ then

$a)$ $\lim_{x \to c} (f(x) \cdot g(x)) = A \cdot B$

$b)$ $\lim_{x \to c} (f(x)/g(x)) = A/B$ if $B \neq 0$

Lemma.

If $\forall \epsilon > 0\, \exists \delta > 0 : 0 < |x - x_0| < \delta \Longrightarrow\; |f(x) - y_0| < M \cdot \epsilon$ for some $M \in \mathbb{R}^+$then

$\forall \epsilon' > 0\, \exists \delta' > 0 : 0 < |x - x_0| < \delta' \Longrightarrow\; |f(x) - y_0| < \epsilon'$

Proof

For a given ϵ' in the last statement let $\epsilon = \epsilon'/M$ in the first statement and then choose δ' for the last statement as the existing δ from the first statement. ■

$\lim_{x \to c} f(x) = A \quad \leftrightarrow \quad \forall \epsilon > 0\, \exists \delta > 0 : 0 < |x - c| < \delta \Longrightarrow\; |f(x) - A| < \epsilon \quad (*)$

$\lim_{x \to c} f(x) = B \quad \leftrightarrow \quad \forall \epsilon > 0\, \exists \delta > 0 : 0 < |x - c| < \delta \Longrightarrow\; |g(x) - B| < \epsilon \quad (**)$

a)

Show: $\forall \epsilon > 0\, \exists \delta > 0 : 0 < |x - c| < \delta \Longrightarrow\; |f(x)g(x) - AB| < \epsilon$

$|f(x)g(x) - AB| = |f(x)g(x) - Bf(x) + Bf(x) - AB| \leq |f(x)(g(x) - B)| + |B(f(x) - A)| \leq$

Pick δ_1 s.t. $|f(x) - A| < 1 \;\Rightarrow\; |f(x)| \leq |A| + 1$

Pick δ_2 s.t. $|g(x) - B| < \epsilon$

Pick δ_3 s.t. $|f(x) - A| < \epsilon$

With $\delta = \min(\delta_1, \delta_2, \delta_3)$ $\;0 < |x - c| < \delta \implies$
$|f(x)g(x) - AB| \leq (|A| + 1)\epsilon + |B|\epsilon \leq (|A| + |B| + 1)\epsilon = M \cdot \epsilon$ for some $M \in \mathbb{R}^+$

Using the lemma:
$\forall \epsilon' > 0 \; \exists \delta' > 0 : 0 < |x - x_0| < \delta' \implies |f(x)g(x) - AB| < \epsilon'$ which means
$\lim_{x \to c}(f(x) \cdot g(x)) = A \cdot B$

b)
$\lim_{x \to c}(f(x) \cdot g(x)) = A \cdot B$ and $\lim_{x \to c}(f(x)/g(x)) = A/B$ with $B \neq 0$

Show $\lim_{x \to c} 1/g(x) = 1/B$, then the rest will follow from the prof of exercise a)
Pick δ_1 s.t. $0 < |x - c| < \delta_1 \;\Rightarrow\; |g(x)| > |B|/2$ (by $(**)$ and $B \neq 0$)
Pick δ_2 s.t. $0 < |x - c| < \delta_2 \;\Rightarrow\; |g(x) - B| < \epsilon$
With $\delta = \min(\delta_1, \delta_2)$ $\;0 < |x - c| < \delta \implies$
$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \left|\frac{B - g(x)}{B \cdot g(x)}\right| \leq \frac{2|B - g(x)|}{|B|^2} = M \cdot \epsilon$ for some $M \in \mathbb{R}^+$

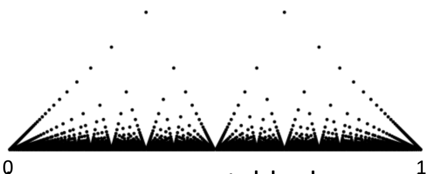
Using the lemma gives $\lim_{x \to c} 1/g(x) = 1/B$ and by the proof of exercise a) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}$

3.22 ?

3.23 Is there a function $f \in C^0(\mathbb{R})$ such that f is continuous on $\mathbb{Q}$ but not on $\mathbb{R} \setminus \mathbb{Q}$?

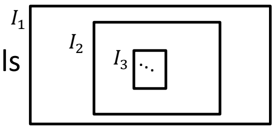
Let $S_c(f)$ be the set of continuity points and look for a function with $S_c(f) = \mathbb{Q}$.
Thomae's function has $S_c(f)$ equal to the set of irrationals, $\mathbb{R} \setminus \mathbb{Q}$

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 1/q \text{ if } x = p/q \text{ in reduced form} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



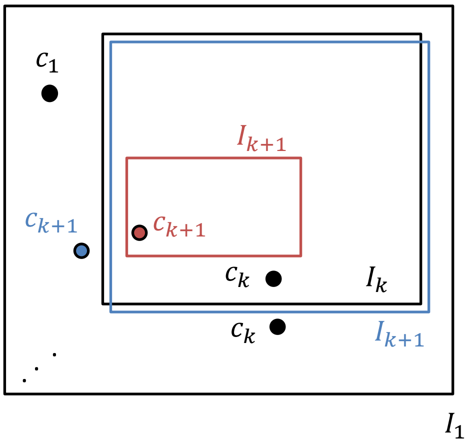
It is much harder to find a function with $S_c(f) = \mathbb{Q}$, continuous on a countable dense set since the disruptive points of discontinuity are more numerous. The irrationals are both dense and uncountable. There is no such function.

Outline of proof:



If $S_c(f)$ is both dense and countable then there will be nested sequence of intervals such that variations of f in I_n tends to zero and $\forall x \in S_c(f) \exists N : n > N \Rightarrow x \notin I_n$.
This implies existence of a limit point $y \in \bigcap_{n=1}^{\infty} I_n$ that must belong to S_c which is a contradiction.

Assume a function s.t. $S_c(f)$ is dense and countable $S_c(f) = \{c_i | i \in \mathbb{Z}^+\}$.
Let $(\epsilon_i)_{i=1}^{\infty}$be a positive sequence $\epsilon_i > 0$ s.t. $\lim_{i\to\infty} \epsilon_i = 0$.
Define inductively a sequence of closed nested intervals $(I_i)_{i=1}^{\infty}$.
From continuity at c_1 we can pick a closed interval I_1 around c_1
s.t. $\forall x \in I_1 : \ |f(x) - f(c_1)| < \epsilon_1$.
When I_k is defined, define I_{k+1} as:
If $c_{k+1} \notin I_k \rightarrow I_{k+1} := I_k$
If $c_{k+1} \in I_k \rightarrow$ Let I_{k+1} be a closed interval s.t.
$c_{k+1} \in I_{k+1}$, $c_k \notin I_{k+1}$ and $\forall x \in I_{k+1} : \ |f(x) - f(c_{k+1})| < \epsilon_{k+1}$



Each I_{n+1} avoids c_n so $\bigcap_{n=1}^{\infty} I_n$ and $S_c(f)$ are disjoint but
as a series of closed and nested intervals $\bigcap_{n=1}^{\infty} I_n$is not empty.
$S_c(f) = \{c_1, c_2, \dots\}$ is a dense set which makes $I_N = I_{N+1} = \cdots$ an impossibility.
If $y_0 \in \bigcap_{n=1}^{\infty} I_n$then it must be a point where f is continuous since:
Given $\epsilon > 0$ we can choose n s.t. $I_n \neq I_{n-1}$ and $\epsilon_n < \epsilon/2$.
$y_0 \in I_n$ so for any $x \in I_n$: $|f(x) - y_0| \leq |f(x) - f(c_n)| + |f(c_n) - y_0| < \epsilon_n + \epsilon_n < \epsilon$
so y_0 is a point where f is continuous but $y_0$is not in the list $c_1, c_2, \dots$ which is a contradiction.
There can be no function $f: \mathbb{R} \rightarrow \mathbb{R}$ where the continuity points are dense and countable, such as $\mathbb{Q}$. ■

3.24 Prove the Archimedean property for $\mathbb{R}$:
There is no positive real pair x, y such that $n \cdot x < y$ for every $n \in \mathbb{N}$.

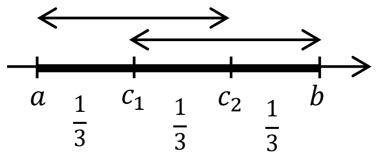
Assume the Archimedean property for $\mathbb{R}$ is false then there is:
$x, y \in \mathbb{R}$ such that $0 < x < y$ and $nx < y$ for every $n \in \mathbb{Z}^+$.
$(nx)_{n=1}^{\infty}$ is upward limited and increasing, it has a least upper bound $M = \sup\limits_{n \in \mathbb{Z}^+}(nx)$.
ϵ-characterization of the supremum with $\epsilon = x > 0$ says there must be an $m \in \mathbb{Z}^+$ s.t.
$mx > M - x$ but then $(m+1)x > M$ contradicts that M is the supremum of $(nx)_{n=1}^{\infty}$.

3.25 Show f continuous on $[a, b] \Rightarrow f$ uniformly continuous on $[a, b]$

Assume f continuous on $[a, b]$

If f is not uniformly continuous on $[a, b]$ then:
$\exists \epsilon_0 > 0\ \forall \delta > 0\ \ \exists x_0, y_0 \in [a, b]$ with $|x_0 - y_0| < \delta$ and $|f(x_0) - f(y_0)| \geq \epsilon_0$ (A)

Split $[a, b]$ into two overlapping intervals
$[a, c_1]$ and $[c_2, b]$ with $c_1 = \frac{2}{3}a + \frac{1}{3}b$ and $c_2 = \frac{1}{3}a + \frac{2}{3}b$.



$a < c_1 < c_2 < b$

If (A) would apply to ~~neither~~ $[a, c_2]$ ~~nor~~ $[c_1, b]$ then
or

$\forall \epsilon\ \exists \delta_0 > 0\ \ \forall x, y \in [\alpha, \beta]$ with $|x - y| < \delta_0$, $|f(x) - f(y)| < \epsilon$
Then with $\epsilon = \epsilon_0$ and $\delta = \min\left(\delta_0, \frac{b-a}{3}\right)$ we would get a contradiction with (A)
so (A) applies to at least one of $[a, c_2]$ and $[c_1, b]$.

Iterating this procedure in a part where uniform continuity does not hold leads to a nested sequence of intervals $I_k = [a_k, b_k]$ with $|f(x_k) - f(y_k)| \geq \epsilon_0 > 0$ for every $k \in \mathbb{Z}^+$ and some $x_k, y_k \in I_k$. (B)

$b_k - a_k = \frac{2}{3}(b_{k-1} - a_{k-1}) \rightarrow \lim\limits_{k\to\infty}(b_k - a_k) = 0$ and $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} b_k = \zeta$ for some $\zeta \in [a, b]$

f is continuous at $\zeta \Longrightarrow \lim\limits_{k\to\infty}(f(x_k) - f(y_k)) = \zeta - \zeta = 0$ which contradicts (B)
$\therefore f$ is uniformly continuous on $[a, b]$ ■

3.26 Assume that $f:[a,b]\to[c,d]$ is continuous and invertible and that f^{-1} is differentiable.
Show that: $\int f^{-1}(y)dy = y\cdot f^{-1}(y) - F\circ f^{-1}(y) + C$
Give the equation a figurative interpretation, a proof without words.

$D(\int f^{-1}(y)dy) = f^{-1}(y).$

$D(y\cdot f^{-1}(y) - F\circ f^{-1}(y) + C) = f^{-1}(y) + y\cdot D\big(f^{-1}(y)\big) - f\circ f^{-1}(y)\cdot D\big(f^{-1}(y)\big) = f^{-1}(y)$

f is continuous and invertible. Assume f increasing $\to f(a)=c$ and $f(b)=d$.
Integrate f^{-1} over $[c,d]$:

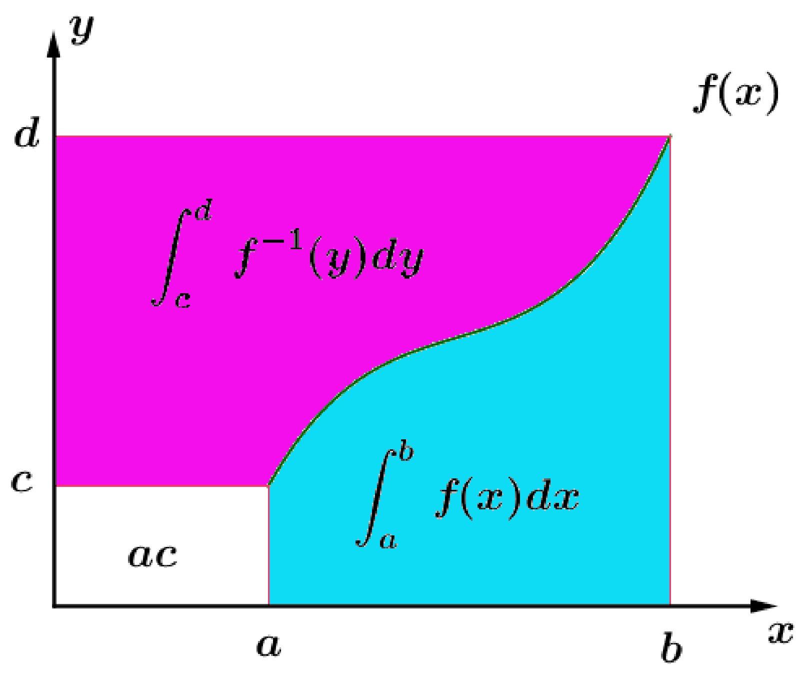
LHS:

$$\int_c^d f^{-1}(y)dy$$

RHS:

$$[y\cdot f^{-1}(y) - F\circ f^{-1}(y)]_c^d = db - ca - \big(F(b)-F(a)\big) = bd - ac - \int_a^b f(x)dx$$

$$\int_a^b f(x)dx + \int_c^d f^{-1}(y)dy = bd - ac$$



The formula for the integral of an inverse function wasn't published until 1905.

3.27 Show that the Cantor function also known as the Devil's staircase c: $[0,1] \to [0,1]$ is increasing, surjective, continuous and has a graph of arc length 2. $c(x)$ is defined by:

1. Express x in base 3 and replace all digits after the first digit=1 (if any) with zeros.
2. Replace all digits=2 after this with digits=1.
3. Reinterpret the sequence as base 2 to get $c(x)$.

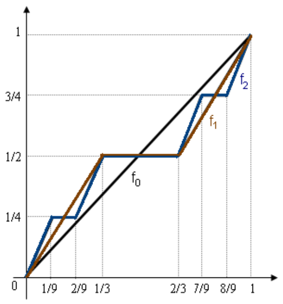
Increasing:
Assume $x < y$. None of the three steps to get $c(x)$ and $c(y)$ can reverse the original order of x and y, at most they can make the results equal so $x < y \Rightarrow c(x) \leq c(y)$.
$c(x)$ is increasing but not strictly increasing.

Surjective:
Any $c(x) = (0.x_1x_2 \dots)_2$ of $[0,1]$ has $x = (0.\tilde{x}_1\tilde{x}_2 \dots)_3$ as a preimage where all $x_i = 1$ has been replaced by digits $\tilde{x}_i = 2$ and the others remain the same.

Continuity:
$c(x)$ can be defined as the limit of a sequence of functions $f_k(x)$ defined by:
$f_0(x) = x$

$$f_{k+1}(x) = \begin{cases} 0.5 \cdot f_k(3x) & \text{if} \quad 0 \leq x \leq 1/3 \\ 0.5 & \text{if} \ \ 1/3 \leq x \leq 2/3 \\ 0.5 + 0.5 \cdot f_k(3x) & \text{if} \ \ 2/3 \leq x \leq 1 \end{cases}$$

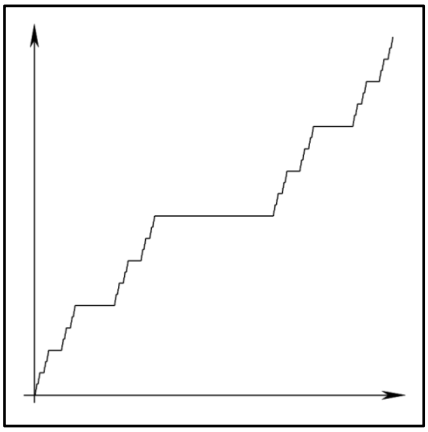
Each f_k is continuous and they converge uniformly to $c(x)$,making it continuous.



$$\max_{x\in[0,1]} |f_{k+1}(x) - f_k(x)| \leq \frac{1}{2} \max_{x\in[0,1]} |f_k(x) - f_{k-1}(x)| \Rightarrow \max_{x\in[0,1]} |c(x) - f_k(x)| \leq \frac{\max_{x\in[0,1]} |f_1(x) - f_0(x)|}{2^{n-1}}$$

Graph of arc has length $L = 2$:
The arc length of a curve $y = f(x)$ with $a \leq x \leq b$ is defined as the supremum of the polygonal arc length based on approximations with partitions $\{a = x_0, x_1, \dots, x_n = b\}$ and segments (x_k, y_k) to (x_{k+1}, y_{k+1}).

Each polygonal segment has $\sqrt{(\Delta x)^2 + (\Delta y)^2} \leq \Delta x + \Delta y$ which for $c(x)$ puts an upper bound $L \leq 2$ since $\sum \Delta x = 1$ and $\sum \Delta y = 1$. To find a polygonal arc with $L \geq 2 - \varepsilon$ note that the sum of constant parts of the Cantor graph approach 1 and their complementing parts with $\sum \Delta y = 1$ are always larger than 1.



If $f: [0,1] \to \mathbb{R}$ is a continuous and increasing function with $f(0) = 0$ and $f(1) = 1$ with curve $\gamma: t \to \big(t, f(t)\big), 0 \leq t \leq 1$:

$$L(\gamma) = 2 \Leftrightarrow f \text{ is a singular function}$$

Singular here means non-constant and continuous on $[0,1]$ with $f'(x) = 0$ almost everywhere, outside a subset of measure zero.

3.28. Show that the area $\int_\alpha^\beta f(x)\,dx$ for $f(x)=1/x$ is unaffected by a rescaling of boundaries $[\alpha,\beta] \curvearrowright [c\alpha, c\beta]$. $\alpha,\beta,c \in \mathbb{R}^+$

$$\int_\alpha^\beta \frac{dx}{x} = [\ln x]_\alpha^\beta = \ln\beta - \ln\alpha$$

$$\int_{c\alpha}^{c\beta} \frac{dx}{x} = [\ln x]_{c\alpha}^{c\beta} = \ln c\beta - \ln c\alpha = \ln c + \ln\beta - \ln c - \ln\alpha = \ln\beta - \ln\alpha$$

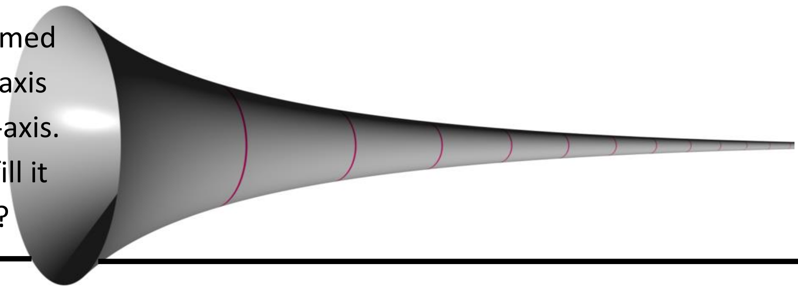
Any function $f(x)$ with this property of invariant area must scale inversely in the y-direction to compensate for the scaling in the x-direction $\rightarrow f(x) \propto 1/x$

$g(x) \in C^1(\mathbb{R}^+)$ can be written $g(x) = \int_1^x f(u)du$ for some f.

If $g(x)$ behaves like a logarithm $g(xy) = g(x) + g(y)$ then $f(x)$ must be of the form $f(x) = C/x$.

$$\int_1^x \frac{C}{x}dx = C\ln x = \frac{\ln x}{\ln e^{1/C}} = \log_{e^{1/C}} x \rightarrow g(x) = \log_a x$$

3.29 Calculate the volume and area formed by rotating $y = 1/x$ around the x-axis for the interval $[1,\infty)$ along the x-axis. How much paint would it take to fill it and how much to paint the inside?



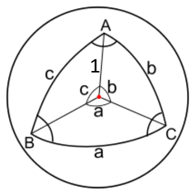
Volume: $\int_1^\infty \pi f(x)^2 dx = \int_1^\infty \frac{\pi}{x^2}dx = \pi[-x^{-1}]_1^\infty = \pi$

Mantel area: $\int_1^\infty 2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}dx = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+x^{-4}}dx > 2\pi\int_1^\infty \frac{dx}{x} = 2\pi[\ln x]_1^\infty = \infty$

It seems that you could fill the form with finite amount of paint but no matter how thin your layer of paint is you could not paint the inside since the area is infinite.

This apparent paradox is known as Toricelli's trumpet or Gabriel's horn.

3.30 Show that the spherical law of cosines $\cos c = \cos a \cos b + \sin a \sin b \cos C$ reduces to the planar law of cosines $c^2 = a^2 + b^2 - 2ab\cos C$ as $a,b,c \rightarrow 0$.



Replace the trigonometric functions of a,b,c with their Taylor series.

$\cos a = 1 - a^2/2 + O(a^4)$ $\quad$ $\sin a = a + O(a^3)$

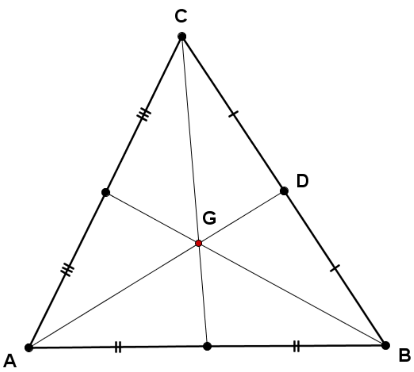
$\cos b = 1 - b^2/2 + O(b^4)$ $\quad$ $\sin b = b + O(b^3)$

$\cos c = 1 - c^2/2 + O(c^4)$ $\quad$ $\sin c = c + O(c^3)$

$1 - c^2/2 + O(c^4) = \left(1 - a^2/2 + O(a^4)\right)\left(1 - b^2/2 + O(b^4)\right) + (a + O(a^3))(b + O(b^3))\cos C \rightarrow$
$c^2 = a^2 + b^2 - 2ab\cos C$ as $a,b,c \rightarrow 0$

3.31 A pyramid has an equilateral triangle as base, the sides are isosceles triangles and the height of the pyramid equals the distance between the height and the base. What is the angle between two sides?

The height in the base plane starts from the intersection of the medians.
The medians intersect for any triangle in a single point which divides the median in proportions 1:2.



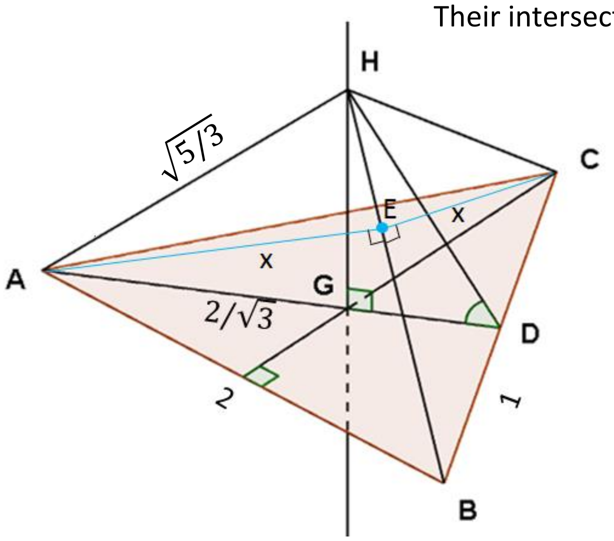
Consider the point formed by averaging over the positions of the vertices.

$$\overrightarrow{\mathrm{OG}} = \frac{1}{3}(x_A + x_B + x_C, y_A + y_B + y_C)$$

$$\overrightarrow{\mathrm{AG}}_x = \overrightarrow{\mathrm{OG}}_x - \overrightarrow{\mathrm{OA}}_x = \frac{1}{3}(x_A + x_B + x_C) - x_A = \frac{1}{3}(-2x_A + x_B + x_C)$$

$$\overrightarrow{\mathrm{AD}}_x = \left(\overrightarrow{\mathrm{AB}} + \tfrac{1}{2}\overrightarrow{\mathrm{BC}}\right)_x = x_B - x_A + \tfrac{1}{2}(x_C - x_B) = \frac{1}{2}(-2x_A + x_B + x_C)$$

$\Rightarrow \overrightarrow{\mathrm{AG}} = \frac{2}{3}\overrightarrow{\mathrm{AD}} \Rightarrow \frac{|\mathrm{AG}|}{|\mathrm{GD}|} = \frac{2}{1}$ This works for all medians and both x and y.
All medians pass through G, the centroid point.
Their intersection divides each median in proportion 2:1.



Assume $|\mathrm{AB}| = 2$ Scale invariance

$|\mathrm{BD}| = \frac{1}{2}|\mathrm{AB}| = 1$

$|\mathrm{GD}| = \frac{1}{3}|\mathrm{AD}| = \frac{1}{3}\sqrt{2^2 - 1^2} = 1/\sqrt{3}$

$|\mathrm{GH}| = |\mathrm{GD}| = 1/\sqrt{3}$ (Picture not drawn to scale)

$|\mathrm{AH}| = \sqrt{\left(2/\sqrt{3}\right)^2 + \left(1/\sqrt{3}\right)^2} = \sqrt{5/3}$

Find the point on BH that makes $\angle\mathrm{AEH}$ and $\angle\mathrm{AEB}$ right angles by introducing α and x.

$|\mathrm{AE}| = x$, $|\mathrm{HE}| = \alpha\sqrt{5/3}$, $|\mathrm{BE}| = (1-\alpha)\sqrt{5/3}$

$$\begin{cases} x^2 + \alpha^2\frac{5}{3} = \frac{5}{3} & (I) \\ x^2 + (1-\alpha)^2\frac{5}{3} = 4 & (II) \end{cases}$$

$(II) - (I) \rightarrow \alpha = -\frac{1}{5} \rightarrow x = \sqrt{8/5}$ (n.b. E lies on the extension of HB)

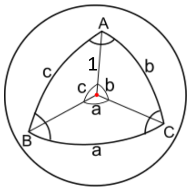
The law of cosines on $\Delta\mathrm{AEC}$ gives: $4 = \frac{8}{5} + \frac{8}{5} - 2\cdot\frac{8}{5}\cos(\angle\mathrm{AEC}) \rightarrow \cos(\angle\mathrm{AEC}) = -\frac{1}{4} \rightarrow \angle\mathrm{AEC} \approx 104°$

```
pA = {0, 0, 0}
pB = {2, 0, 0}
pC = {1, Sqrt[3], 0}
pH = {1, 1/Sqrt[3], 1/Sqrt[3]}
AB = pB - pA
AH = pH - pA
BC = pC - pB
BH = pH - pB
```

```
nABH = Normalize[Cross[AB, AH]]
nBCH = Normalize[Cross[BC, BH]]
nABH.nBCH
1
-
4
N[180 / Pi * ArcCos[1 / 4]]
75.5225
180 - %
104.478
```

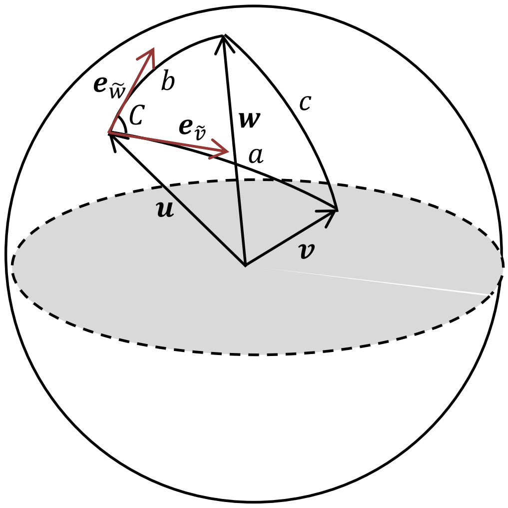
Checking calulation with Mathematica based on coordinates of points and normals calculated from vector products.

3.32 Prove the spherical law of cosines
$\cos c = \cos a \cos b + \sin a \sin b \cos C.$



Solve by using vectors and scalar products $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| \cdot \cos \alpha$.
Let the points A, B and C on the unit sphere be given by unit vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$.

$\cos a = \boldsymbol{u} \cdot \boldsymbol{v}$
$\cos b = \boldsymbol{u} \cdot \boldsymbol{w}$
$\cos c = \boldsymbol{v} \cdot \boldsymbol{w}$



$\cos C = \boldsymbol{e}_{\tilde{v}} \cdot \boldsymbol{e}_{\tilde{w}}$ $\quad$ $\boldsymbol{e}_{\tilde{v}}, \boldsymbol{e}_{\tilde{w}}$ in the tangential plane at C
$\boldsymbol{e}_{\tilde{v}}$ in (u,v)-plane and $\boldsymbol{e}_{\tilde{w}}$ in (u,w)-plane.

Project $\boldsymbol{v}$ along $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$-plane spanned by $\boldsymbol{e}_{\tilde{v}}, \boldsymbol{e}_{\tilde{w}}$.

$\boldsymbol{v}_u = (\boldsymbol{v} \cdot \boldsymbol{u})\boldsymbol{u} = \cos a \cdot \boldsymbol{u}$
$\boldsymbol{v}_{u^\perp} = \boldsymbol{v} - \boldsymbol{v}_u = \boldsymbol{v} - \cos a \cdot \boldsymbol{u} \quad \boldsymbol{v}_{u^\perp} \parallel \boldsymbol{e}_{\tilde{v}}$

$$\boldsymbol{e}_{\tilde{v}} = \frac{\boldsymbol{v}_{u^\perp}}{\|\boldsymbol{v}_{u^\perp}\|} = \frac{\boldsymbol{v} - \cos a \cdot \boldsymbol{u}}{\sin a}$$
$$\boldsymbol{e}_{\tilde{w}} = \frac{\boldsymbol{w}_{u^\perp}}{\|\boldsymbol{w}_{u^\perp}\|} = \frac{\boldsymbol{w} - \cos b \cdot \boldsymbol{u}}{\sin b}$$

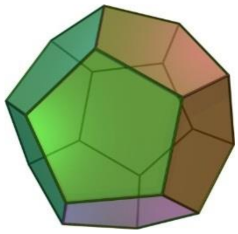
$$\cos C = \boldsymbol{e}_{\tilde{v}} \cdot \boldsymbol{e}_{\tilde{w}} = \frac{1}{\sin a \sin b} (\boldsymbol{v} - \cos a \cdot \boldsymbol{u}) \cdot (\boldsymbol{w} - \cos b \cdot \boldsymbol{u})$$
$$\cos C = \frac{\cos c - \cos b \cos a - \cos a \cos b + \cos a \cos b}{\sin a \sin b} = \frac{\cos c - \cos b \cos a}{\sin a \sin b}$$

$$\sin a \sin b \cos C = \cos \mathrm{c} - \cos a \cos b$$

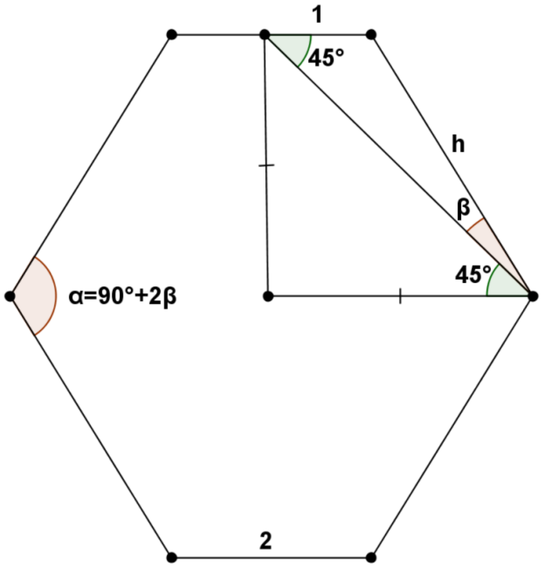
$$cos\ c = \cos a \cos b - \sin a \sin b \cos C$$

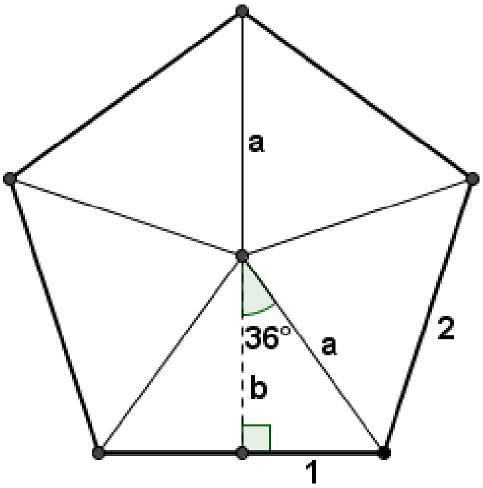
$C = 90°$ gives a generalization of Pythagoras theorem on a sphere, $cos\ c = \cos a \cos b$.

3.33 Calculate the inner angle between adjacent faces in a regular dodecahedron bounded by 12 pentagons. (Dodecahedron from Greek, meaning 12 faces)



Let us assume that the edges of the dodecahedron are two units long.
The cross-section looks like the left picture where **h** is the height in the pentagon and **α** is the angle we are looking for.





$\sin 36° = 1/a$

$\tan 36° = 1/b$

$$h = a + b = \frac{1}{\sin 36°} + \frac{1}{\tan 36°} = \frac{1 + \cos 36°}{\sin 36°}$$

Using the law of sines on the triangle containing angle β:

$$\frac{\sin 45°}{h} = \frac{\sin \beta}{1} \Rightarrow \beta = \arcsin\left(\frac{\sin 36°}{\sqrt{2}(1 + \cos 36°)}\right)$$

$$\alpha = 90° + 2\arcsin\left(\frac{\sin 36°}{\sqrt{2}(1 + \cos 36°)}\right) \approx 116.57°$$

According to Wikipedia the angle between the two faces equals $2\arctan\varphi$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Let us see if we can prove that:

$$90° + 2\arcsin\underbrace{\left(\frac{\sin 36°}{\sqrt{2}(1+\cos 36°)}\right)}_{x} = 2\arctan\frac{1+\sqrt{5}}{2}$$

$$\boxed{\tan(90°+\alpha) = -\frac{1}{\tan\alpha}} \qquad \boxed{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}$$

$\tan(90° + 2\arcsin x) = -\frac{1}{\tan(2\arcsin x)}$ $\qquad$ $\tan(2\arctan\varphi) = \frac{2\varphi}{1-\varphi^2} = \frac{1+\sqrt{5}}{1-(6+2\sqrt{5})/4} = -2$

$$\boxed{\tan\arcsin\alpha = \frac{\alpha}{\sqrt{1-\alpha^2}}}$$

$$\tan(2\arcsin x) = \frac{2\tan\arcsin x}{1-\tan^2\arcsin x} = \frac{2x/\sqrt{1-x^2}}{1-x^2/(1-x^2)} = \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

where: $x = \frac{\sin 36°}{\sqrt{2}(1+\cos 36°)}$, $\cos 36° = \frac{1+\sqrt{5}}{4}$, $\sin 36° = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$ (Proof given in exercise x.x)

Remains to prove: $\frac{2x\sqrt{1-x^2}}{1-2x^2} = \frac{1}{2}$

Squaring both sides gives: $20x^4 - 20x^2 + 1 = 0$

where: $x = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}\left(1+\frac{1+\sqrt{5}}{4}\right)} = \frac{\sqrt{5-\sqrt{5}}}{5+\sqrt{5}} \implies x^2 = \frac{5-\sqrt{5}}{30+10\sqrt{5}}$

Remains to prove: $20y^2 - 20y + 1 = 0$

where: $y = \frac{1}{10}\left(\frac{5-\sqrt{5}}{3+\sqrt{5}}\right) \Rightarrow y^2 = \frac{1}{20}\left(\frac{3-\sqrt{5}}{7+3\sqrt{5}}\right) \Rightarrow 20y^2 - 20y + 1 =$

$$\frac{3-\sqrt{5}}{7+3\sqrt{5}} - \frac{10-2\sqrt{5}}{3+\sqrt{5}} + 1 =$$

$$\frac{(9-5)-(70-14\sqrt{5}+30\sqrt{5}-30)}{21+7\sqrt{5}+9\sqrt{5}+15} + 1 =$$

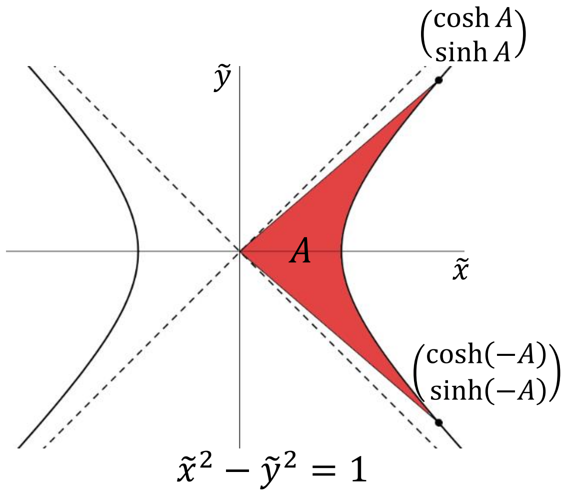
$$\frac{-36-16\sqrt{5}}{36+16\sqrt{5}} + 1 = 0$$

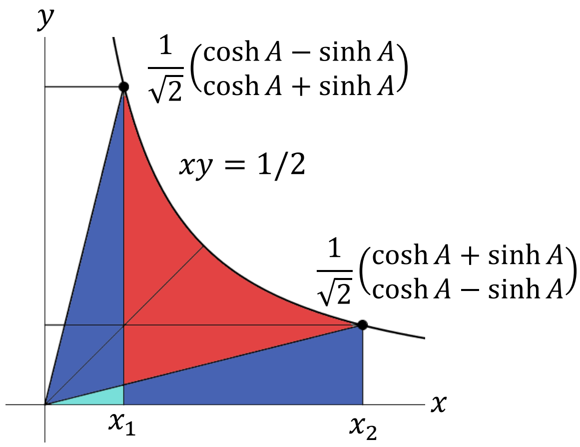
This concludes the proof and in the process we got another property of the golden ratio.

$$\boxed{\begin{array}{c}\text{Golden ratio } \varphi \\ 2 + \tan(2\arctan\varphi) = 0\end{array}}$$

3.34 Use the definition of the hyperbolic functions from a hyperbola to show

$$\cosh A = \tfrac{1}{2}(e^{A} + e^{-A}) \qquad \operatorname{arcosh} t = \ln\left(t + \sqrt{t^2 - 1}\right)$$
$$\sinh A = \tfrac{1}{2}(e^{A} - e^{-A}) \qquad \operatorname{arsinh} t = \ln\left(t + \sqrt{t^2 + 1}\right)$$





Rote the unit hyperbola 45°, $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}$ $\qquad x_1 = (\cosh A - \sinh A)/\sqrt{2}$, $\; x_2 = (\cosh A + \sinh A)/\sqrt{2}$

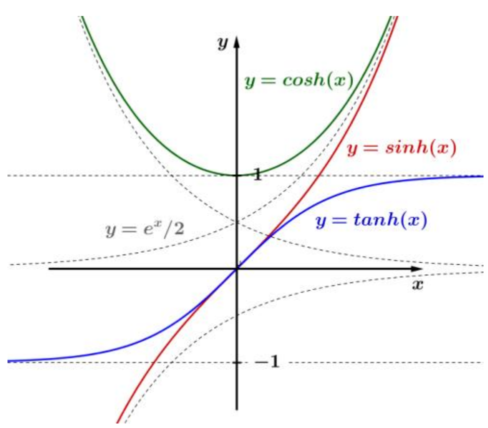
$$A = \int_{x_1}^{x_2} \frac{dx}{2x} = \frac{1}{2}[\ln x]_{x_1}^{x_2} = \frac{1}{2}\ln\frac{x_2}{x_1} = \frac{1}{2}\ln\left(\frac{\cosh A + \sinh A}{\cosh A - \sinh A}\right) = \frac{1}{2}\ln\left(\frac{(\cosh A + \sinh A)^2}{\cosh^2 A - \sinh^2 A}\right)$$

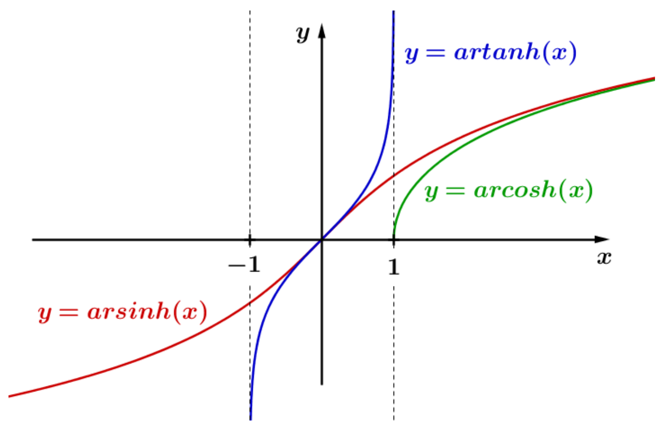
$$A = \ln(\cosh A + \sinh A)$$
$$e^{A} = \cosh A + \sinh A$$
$$e^{-A} = \frac{1}{\cosh A + \sinh A} = \frac{\cosh A - \sinh A}{\cosh^2 A - \sinh^2 A} = \cosh A - \sinh A$$

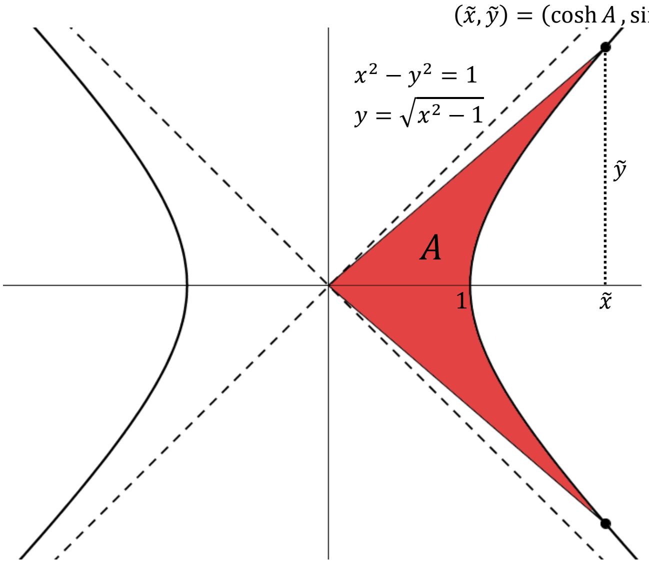
$$e^{A} + e^{-A} = 2\cosh A \rightarrow \quad \cosh A = \frac{e^{A} + e^{-A}}{2}$$
$$e^{A} - e^{-A} = 2\sinh A \rightarrow \quad \sinh A = \frac{e^{A} - e^{-A}}{2}$$

$A = \ln(\tilde{x} + \tilde{y})$ with $\tilde{x}^2 - \tilde{y}^2 = 1 \qquad \tilde{x} = \sqrt{\tilde{y}^2 + 1}$, $\; \tilde{y} = \sqrt{\tilde{x}^2 - 1}$

$$\tilde{x} = \cosh A = \cosh\left(\ln\left(\tilde{x} + \sqrt{\tilde{x}^2 - 1}\right)\right) \qquad \operatorname{arcosh} t = \ln\left(t + \sqrt{t^2 - 1}\right)$$
$$\rightarrow$$
$$\tilde{y} = \sinh A = \sinh\left(\ln\left(\sqrt{\tilde{y}^2 + 1} + \tilde{y}\right)\right) \qquad \operatorname{arsinh} t = \ln\left(t + \sqrt{t^2 + 1}\right)$$





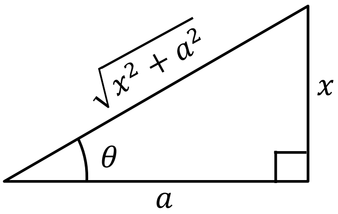


Solutions to mathematical problems are never unique. The legendary mathematician Paul Erdős (1913-1996) often talked about "The Book" where God kept one solution, the most natural and elegant solution to each mathematical problem. The previous solution might qualify as such a solution. A solution that would not make it to the book is the following one, but it is a good illustration of various techniques of integration.

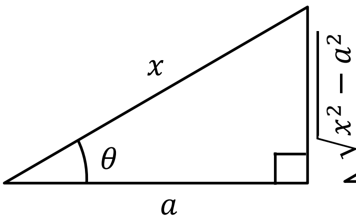
Calculate area A, this time without a 45° rotation.

$$A = 2\left(\frac{\tilde{x}\tilde{y}}{2} - \int_1^{\tilde{x}} \sqrt{x^2 - 1}dx\right)$$

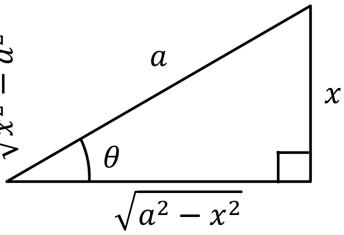
Integrals containing $x^2 + a^2$, $x^2 - a^2$ or $a^2 - x^2$ can often be solved by trigonometric substitution. (Hyperbolic substitution are suitable for $\sqrt{x^2 - 1}$ but it would become circular in this case.)



$$x = a \tan\theta$$
$$dx = a\sec^2\theta$$
$$\sqrt{x^2 + a^2} = a\sec\theta$$



$$x = a\sec\theta$$
$$dx = a\sec\theta\tan\theta$$
$$\sqrt{x^2 - a^2} = a\tan\theta$$



$$x = a\sin\theta$$
$$dx = a\cos\theta$$
$$\sqrt{a^2 - x^2} = a\cos\theta$$

Integration by substitution

$$\int_{x_a}^{x_b} f(x)dx \left\{\begin{matrix} x = g(t) \\ dx = g'(t)dt \\ x_a = g(t_a) \\ x_b = g(t_b) \end{matrix}\right\} = \int_{t_a}^{t_b} f\big(g(t)\big)g'(t)dt$$

$$\int_1^{\tilde{x}} \sqrt{x^2 - 1}dx \left\{\begin{matrix} x = \sec\theta \\ dx = \sec\theta\tan\theta \\ 1 = \sec 0 \\ \tilde{x} = \sec\tilde{\theta} \end{matrix}\right\} = \int_0^{\tilde{\theta}} \tan^2\theta\sec\theta\, d\theta = \int_0^{\tilde{\theta}} (\sec^3\theta - \sec\theta)d\theta$$

$\int \sec^3\theta\, d\theta$ looks like a candidate for integration by a reduction formula. Such formulas look something like $I_n \equiv \int f(n,x)\,dx = g(I_k)$ with $k < n$ where $f(n,x)$ contains a part that is raised to the power n which get reduced to a lower power in I_k. They are often derived with integration by parts (IBP).

Integration by parts

$$\int (u'v + uv')dx = uv \to \int_a^b f(x)\,g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

Examples of <u>integration by reduction formulae</u>:

$$I_n = \int \cos^n x\,dx \quad \to \quad I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

$$I_n = \int x^n e^{ax} dx \quad \to \quad I_n = \frac{1}{a}(x^n e^{ax} - n I_{n-1})$$

$$I_n = \int \frac{x^n}{\sqrt{ax+b}} dx \quad \to \quad I_n = \frac{2x^n\sqrt{ax+b}}{a(2n+1)} - \frac{2nb}{a(2n+1)} I_{n-1}$$

$$I_{n,m} = \int \frac{dx}{x^m(x^2+a^2)^n} \quad \to \quad I_{n,m} = a^{-1}\big(I_{m,n-1} - I_{m-2,n}\big)$$

$\sec\theta = 1/\cos\theta$
$\tan^2\theta = \sec^2\theta - 1$
$D(\sec\theta) = \tan\theta\sec\theta$
$D(\tan\theta) = \sec^2\theta$

$$I_n = \int \sec^n\theta\,d\theta = \int \sec^2\theta\sec^{n-2}\theta = \tan\theta\sec^{n-2}\theta - (n-2)\int(\sec^n\theta - \sec^{n-2}\theta)d\theta \to$$
$$(n-1)I_n = \tan\theta\sec^{n-2}\theta + (n-2)I_{n-2}$$

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int \sec\theta\,d\theta$$
$$\int \sec\theta\,d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta\,d\theta}{1-\sin^2\theta}\begin{Bmatrix} u = \sin\theta \\ du = \cos\theta\,d\theta \end{Bmatrix} = \int \frac{du}{1-u^2} = \frac{1}{2}\int\left(\frac{1}{1+u} - \frac{1}{1-u}\right)du =$$

The last part was obtained by <u>partial fraction decomposition</u> which is used when integrating rational functions.

If f and g are non-zero polynomials over a field K with $g = \prod_{i=1}^{n} {P_i}^{n_i}$ written as a product of distinct irreducible polynomials. (For $K = \mathbb{R}$ this means P_i is a polynomial of degree 1 or 2.)

$$\frac{f}{g} = \sum_{i=1}^{k}\sum_{j=i}^{n_i} \frac{a_{ij}}{{P_i}^j} \qquad \begin{array}{l}\text{With unique polynomials } b \text{ and } a_{ij}\ \deg a_{ij} < \deg P_i \\ \text{If } \deg f < \deg g \text{ then } b = 0\end{array}$$

$$\int \sec\theta\,d\theta = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin\theta)^2}{1-\sin^2\theta}\right| = \ln|\sec\theta + \tan\theta|$$

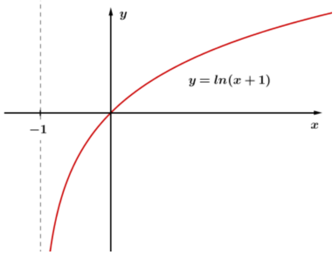
$$\int_1^{\tilde{x}} \sqrt{x^2-1}dx = \frac{1}{2}\tan\tilde{\theta}\sec\tilde{\theta} - \frac{1}{2}\ln|\sec\tilde{\theta} + \tan\tilde{\theta}| = \frac{\tilde{x}}{2}\underbrace{\sqrt{\tilde{x}^2-1}}_{\tilde{y}} - \frac{1}{2}\ln\left|\tilde{x} + \sqrt{\tilde{x}^2-1}\right|$$

$$A = \tilde{x}\tilde{y} - 2\int_1^{\tilde{x}}\sqrt{x^2-1}dx = \ln|\tilde{x}+\tilde{y}| = \ln|\cosh A + \sinh A| \to (\text{ See first solution })$$

$$\cosh x = \frac{e^x+e^{-x}}{2} \qquad \operatorname{arcosh} x = \ln\big(x + \sqrt{x^2-1}\big)$$
$$\sinh x = \frac{e^x-e^{-x}}{2} \qquad \operatorname{arsinh} x = \ln\big(x + \sqrt{x^2+1}\big)$$

3.35 Derive the Taylor series expansions of $\ln(x+1)$, $\arctan x$ and $\operatorname{artanh} x$ around $x = 0$ and show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

$f(x) = \ln(x+1) \quad f \in C^\infty((-1,\infty),\mathbb{R})$
$\mathrm{D}(\ln(x+1)) = (x+1)^{-1}$
$\mathrm{D}^n(\ln(x+1)) = (-1)^{n-1}(n-1)!\,(x+1)^{-n} \quad n \geq 1$



Taylor's theorem

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + R_n(x) \;\rightarrow\; \ln(x+1) = \sum_{k=1}^{n-1} (-1)^{k-1} \frac{x^k}{k} + R_n(x)$$

$$R_n(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^n(t)dt = \frac{f^{(n)}(\xi)}{n!} x^n = \frac{(-1)^{n-1}}{n(\xi+1)^n} x^n \quad (\xi \in [0,x])$$

Case 1: $0 \leq x \leq 1$

$$\xi + 1 \geq 1 \Rightarrow |R_n(x)| \leq \frac{x^n}{n} \leq \frac{1}{n} \Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

Case 2: $-1 < x < 0$

$$|R_n(x)| = \int_x^0 \frac{(t-x)^{n-1}}{(t+1)^n} dt = \int_x^0 \left(\frac{t-x}{t+1}\right)^{n-1} \cdot \frac{1}{t+1} dt \leq |x|^{n-1} \int_x^0 \frac{dt}{x+1} \leq \frac{|x|^n}{x+1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$g(t) = \frac{t-x}{t+1} = 1 - \frac{1+x}{t+1} \text{ increases with } t \;\Rightarrow \frac{t-x}{t+1} \leq -x = |x|$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for} \quad x \in (-1,1]$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$\boxed{\begin{aligned} &y = \arctan x \quad x = \tan y = \frac{\sin y}{\cos y} \quad D\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2} \\ &\frac{dy}{dx} = \frac{1}{dx/dy} = \cos^2 y = \frac{\cos^2 y}{\cos^2 y + \sin^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \end{aligned}}$$

$f(x) = \arctan x \quad f \in C^\infty(\mathbb{R},\mathbb{R})$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} =$$

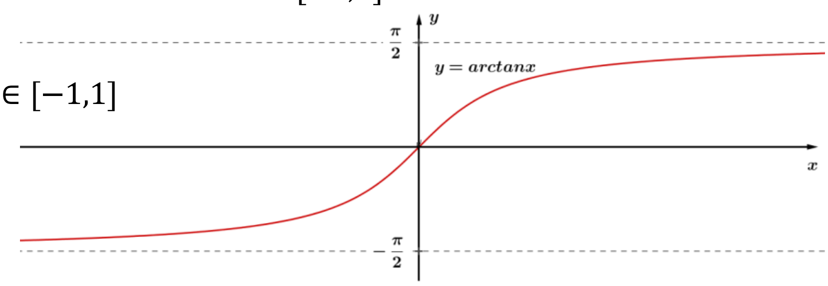
$$\boxed{\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x} \begin{pmatrix} x \neq 1 \\ x = -t^2 \\ t \in \mathbb{R} \end{pmatrix} \rightarrow \frac{1}{1+t^2} = \sum_{k=0}^{n-1} (-1)^k t^{2k} + (-1)^n \frac{t^{2n}}{1+t^2}}$$

$$\int_0^x 1 - t^2 + t^4 - \cdots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + r_n$$

$$|r_n| = \int_0^x \frac{t^{2n}}{1+t^2} dt \leq \int_0^x t^{2n} dt = \frac{x^{2n+1}}{2n+1} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for } x \in [-1,1]$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \text{ for } x \in [-1,1]$$

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$



A Taylor series is uniquely determined by f with x^k-coefficient $f^k(0)/k!$.

$f(x) = \arctan x \to f^{(2k)}(0) = 0$ and $f^{(2k+1)}(0) = (-1)^k(2k)!$

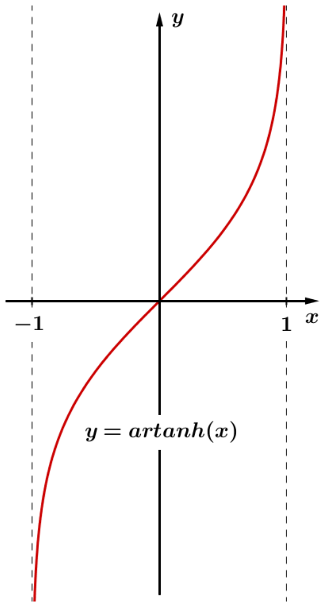
$f(x) = \operatorname{artanh} x \quad f \in C^\infty((-1,1),\mathbb{R})$

$$\operatorname{artanh} x = \int_0^x \frac{dt}{1-t^2} =$$

$$y = \operatorname{artanh} x \quad x = \tanh y = \frac{\sinh y}{\cosh y} \quad \begin{array}{l} \cosh^2 x - \sinh^2 x = 1 \\ D(\sinh x) = \cosh x \\ D(\cosh x) = \sinh x \end{array}$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{\cosh^2 y}{\cosh^2 y - \sinh^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$$

$$\sum_{k=0}^{n-1} x^k = \frac{1-x^n}{1-x}\begin{pmatrix} x \neq 1 \\ x = t^2 \\ t \neq 1 \end{pmatrix} \to \frac{1}{1-t^2} = \sum_{k=0}^{n-1} t^{2k} + \frac{t^{2n}}{1-t^2}$$



$$\int_0^x 1 + t^2 + t^4 + \cdots + t^{2n-2} + \frac{t^{2n}}{1-t^2}dt = x + \frac{x^3}{3} + \cdots + \frac{x^{2n-1}}{2n-1} + r_n$$

$$|r_n| = \int_0^x \frac{t^{2n}}{1-t^2}dt \leq x^{2n}\int_0^x \frac{dt}{1-t^2} = x^{2n} \operatorname{artanh} x \ \to 0 \text{ as } n \to \infty\,, |x| < 1$$

$$\operatorname{artanh} x = \ x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \text{ for } x \in (-1,1)$$

$$\operatorname{artanh}(ix) = i\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}\right) = i \arctan x$$

A consequence of trigonometric functions being based upon $x^2 + y^2 = 1$
whereas hyperbolic functions are based upon $x^2 - y^2 = x^2 + (iy)^2 = 1$

The Taylor expansions of $\ln(x+1)$, $\arctan x$ and $\operatorname{artanh} x$ are of little computational value since they have a slow and limited area of convergence.

A direct calculation of higher derivatives of $\arctan x$ can be done inductively.
$D^n(\arctan x) = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$ with $\sin\theta = 1/\sqrt{1+x^2}$
This leads to an alternative Taylor formula:

$$\frac{\pi}{2} - \theta = \sum_{n=1}^{\infty}\frac{1}{n}\cos^n\theta\sin n\theta \to \pi = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right)$$

Another formula of this type was discovered in 1995 by Bailey-Borwein-Plouffe

$$\pi = \sum_{n=0}^{\infty}\frac{1}{16^n}\left(\frac{2}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)$$

This formula has fast convergence and it can be used for calculating the n-th digit of π in a hexadecimal base without calculating preceding digits.
This was a big surprise at the time.

3.36 Calculate $f_\omega(3)$ and show that $f_{\omega^2}(n) > n \to \cdots \to n$ *(n n's)*

f_α comes from the fast-growing hierarchy.

$$f_0(n) = n+1$$
$$f_{\alpha+1}(n) = f_\alpha^n(n)$$
$$f_\alpha(n) = f_{\alpha_n}(n) \text{ when } \alpha = \lim_n \alpha_n \text{ is a limit ordinal.}$$

$f_0(n) = n+1$
$f_1(n) = f_0^n(n) = n + n = 2n$
$f_2(n) = f_1^n(n) = n2^n$

$f_\omega(3) = f_3(3) = f_2^3(3) = 3\cdot 2^3 \cdot \left(3\cdot 2^3 \cdot 2^{3\cdot 2^3}\right) \cdot \left(3\cdot 2^3 \cdot \left(3\cdot 2^3 \cdot 2^{3\cdot 2^3}\right)\cdot 2^{3\cdot 2^3\cdot 2^{3\cdot 2^3}}\right) =$
$3^3 \cdot 2^{3\cdot 3 + 2\cdot 3\cdot 2^3 + 3\cdot 2^3 \cdot 2^{3\cdot 2^3}} = 2^{402653241}\cdot 3^3$

$f_0(n) = n+1$
$f_1(n) = f_0^n(n) = n + n = 2n$
$f_2(n) = f_1^n(n) = 2^n \cdot n > 2^{n+1} > 2^n$
$f_3(n) = f_2^n(n) = 2\uparrow(2\uparrow\cdots(2\uparrow n)\ldots)\ \ n\ 2's > \ 2\uparrow\uparrow(n+1) > 2\uparrow\uparrow n$
$\vdots$
$f_k(n) > 2\uparrow^{k-1}(n+1)\ \ (n>2)$
$f_\omega(n) = f_n(n) > 2\uparrow^{n-1}(n+1)$

Show $p\to q\to r = p\uparrow^r q \quad p,q,r \in \mathbb{Z}^+$

$p\to q\to 1 = p\to q = p^q = p\uparrow^1 q$

Assume $p\to q\to r = p\uparrow^r q$
$p\to q\to(r+1) = p\to(p\to\cdots\to(p\to(p)\to r)\to\cdots\to r)\to r) \quad q\ p's =$
$= p\uparrow^r\left(\ldots\left(p\uparrow^r(p\uparrow^r p)\right)\ldots\right) \quad q\ \ p's =$
$= p\uparrow^{r+1} q$
$\therefore p\to q\to r = p\uparrow^r q \quad p,q,r\in\mathbb{Z}^+$

Show $X\to n\to k \equiv g_{X,k}(n) \Longrightarrow X\to n\to(k+1) = g_{X,k}^n(1) \quad X$ is a Conway chain, $n,k\in\mathbb{Z}^+$.

Example:
$g_{X,k}(1) = X\to 1\to k = (X) \qquad (X)$ means evaluate Conway chain.
$g_{X,k}(2) = X\to 2\to k = X\to(X)\to k = g_{X,k}\big((X)\big) = g_{X,k}^2(1)$

Let $g_{X,k}(n) = X\to n\to k$
$X\to n\to(k+1) =$

$$\overbrace{X\to(X\to(\ldots\underbrace{(X\to(X)\to k)}_{g_{X,k}^2(1)}\to\cdots\to k))}^{g_{X,k}^n(1)} \quad n\ X's)$$

$$f_\omega(n) = f_n(n) > 2 \uparrow^{n-1} (n+1) = \underbrace{2 \to (n+1)}_{X} \to \underbrace{(n-1)}_{\tilde{n}} \to 1 \quad (n > 2)$$

$$f_\omega(n) > X \to \tilde{n} \to 1 \equiv g_{X,1}(\tilde{n}) \qquad n = \tilde{n} + 1 \;\;,\;\; g_{X,1}(y) < f_\omega(1+y)$$

$$X \to \tilde{n} \to 2 = g_{X,1}^{\tilde{n}}(1) < g_{X,1}^{n-2}\big(f_\omega(2)\big) < g_{X,1}^{n-3}(f_\omega(1 + f_\omega(2))) < g_{X,1}^{n-3}\big(f_\omega^2(3)\big) < \cdots < f_\omega^{n-1}(n)$$
$$X \to \tilde{n} \to 2 < f_\omega^{n-1}(n) < f_\omega^{n}(n) = f_{\omega+1}(n)$$

$$f_{\omega+1}(n) > 2 \to (n+1) \to (n-1) \to 2$$

Assume $f_{\omega+k}(n) > \underbrace{2 \to (n+1)}_{X} \to \underbrace{(n-1)}_{\tilde{n}} \to \underbrace{k+1}_{\tilde{k}} \equiv g_{X,\tilde{k}}(\tilde{n}) \qquad g_{X,\tilde{k}}(y) < f_{\omega+k}(y+1)$

$$X \to \tilde{n} \to \big(\tilde{k}+1\big) = g_{X,\tilde{k}}^{\tilde{n}}(1) < g_{X,\tilde{k}}^{n-2}\big(f_{\omega+k}(2)\big) < g_{X,k}^{n-3}(f_{\omega+k}(f_{\omega+k}(2)+1) <$$
$$< g_{X,k}^{n-3}\Big(f_{\omega+k}^2(3)\Big) < \cdots < f_{\omega+k}^{n-1}(n) < f_{\omega+k}^{n}(n) = f_{\omega+k+1}(n)$$

$$\therefore f_{\omega+k}(n) > 2 \to (n+1) \to (n-1) \to (k+1)$$

$$f_{\omega 2}(n) = f_{\omega+n}(n) > \underbrace{2 \to (n+1) \to (n-1)}_{X} \to \underbrace{(n+1)}_{\tilde{n}} \to \underbrace{1}_{\tilde{k}}$$

Assume $f_{\omega 2+k}(n) > X \to \tilde{n} \to \tilde{k} \equiv g_{X,\tilde{k}}(\tilde{n}) \qquad \tilde{n} = n+1\,, \tilde{k} = k+1\,,\; g_{X,\tilde{k}}(y) < f_{\omega 2+k}(y-1)$

$$X \to \tilde{n} \to \big(\tilde{k}+1\big) = g_{X,\tilde{k}}^{\tilde{n}}(1) < g_{X,\tilde{k}}^{\tilde{n}-1}\big(f_{\omega 2+k}(0)\big) < g_{X,\tilde{k}}^{\tilde{n}-1}\big(f_{\omega 2+k}(1)\big) < g_{X,\tilde{k}}^{\tilde{n}-2}(f_{\omega 2+k}(f_{\omega 2+k}(1)-1))) <$$
$$g_{X,\tilde{k}}^{\tilde{n}-2}\Big(f_{\omega 2+k}^2(1)\Big) < \cdots < f_{\omega 2+k}^{\tilde{n}}(1) < f_{\omega 2+k}^{n+1}(n+1) = f_{\omega 2+k+1}(n).$$

$$\therefore f_{\omega 2+k}(n) > 2 \to (n+1) \to (n-1) \to (n+1) \to (k+1)$$

$$f_{\omega 3}(n) = f_{\omega 2+n}(n) > 2 \to (n+1) \to (n-1) \to (n+1) \to (n+1)$$

$\vdots$

$$f_{\omega k}(n) > 2 \to (n+1) \to (n-1) \to (n+1) \to (n+1) \to \cdots \to (n+1) \quad (k+1)\, n's$$

$$f_{\omega^2}(n) = f_{\omega n}(n) > 2 \to (n+1) \to (n-1) \to (n+1) \to (n+1) \to \cdots \to (n+1) \quad (n+1)\, n's$$

$$f_{\omega^2}(n) > n \to^{(n+1)} n$$

3.37 A real or complex series $\sum_{k=0}^{\infty} a_k$ is said to be absolutely convergent if $S_n = \sum_{k=0}^{n}|a_k|$ is limited ($\sum_{k=0}^{\infty}|a_k| = \sup\{S_n | n \in \mathbb{N}_0\} = S$).
A series $\sum_{k=0}^{\infty} b_k$ that is convergent $\left(\lim_{n\to\infty}(\sum_{k=0}^{n} b_k) \in \mathbb{C}\right)$ without being absolutely convergent is conditionally convergent. Show that:

I. Absolute convergence $\Rightarrow$ convergence.
II. The sum of absolutely convergent series is independent of the ordering order of the terms.
III. The sum of a real conditionally convergent series can attain any real number with an appropriate summation order.

I.
Let $\sum c_k$ be an absolutely convergent series with $c_k = a_k + ib_k$, ($a_k, b_k \in \mathbb{R}$, $S_n = \sum_{k=0}^{n} c_k$).

$$\sum_{k=0}^{\infty}|c_k| < \infty \Rightarrow \lim_{n\to\infty}\sum_{k=n}^{\infty}|c_k| = 0 \Rightarrow \lim_{n\to\infty}\sum_{k=n}^{\infty}|a_k| = 0 \wedge \lim_{n\to\infty}\sum_{k=n}^{\infty}|b_k| = 0 \quad \begin{pmatrix} 0 \le |a_k| \le |c_k| \\ 0 \le |b_k| \le |c_k| \end{pmatrix}$$

The partial sum of the real parts $A_n = \sum_{k=0}^{n} a_k$ is a Cauchy sequence:

$$0 \le |A_m - A_n| \le \left|\sum_{k=n+1}^{m} a_k\right| \le \left(\sum_{k=n+1}^{m} |a_k|\right) \to 0 \text{ as } m,n \to \infty \quad (\text{Assume } m \ge n)$$

The same goes for partial sums of imaginary parts $B_n = \sum_{k=0}^{n} b_k$ and $m < n$.

A complete matric space M is called complete if every Cauchy sequence of points in M has a limit in M. The real numbers can be constructed as equivalence classes of Cauchy sequences in $\mathbb{Q}$ which makes $\mathbb{R}$ a complete space by construction.

$$\begin{matrix} \lim_{n\to\infty} A_n = A \in \mathbb{R} \\ \lim_{n\to\infty} B_n = B \in \mathbb{R} \end{matrix} \quad S = A + iB \Rightarrow 0 \le |S - S_n| \le |A - A_n| + |B - B_n| \to 0 \text{ as } n \to \infty$$

$\lim_{n\to\infty} S_n = S \in \mathbb{C}$ *means that* $\sum_{k=0}^{\infty} c_k$ is a convergent series.

II.
Let $\sum_{k=0}^{\infty} a_k$ be an absolutely convergent series with limit S and $\sigma: \mathbb{N}_0 \to \mathbb{N}_0$ a bijection that rearranges terms to a new series with $b_k = a_{\sigma(k)}$.

For any given $\varepsilon > 0 \exists N: n \ge N \Rightarrow \begin{cases} |S - \sum_{k=0}^{n} a_k| < \varepsilon/2 \\ \sum_{k\ge n}|a_k| < \varepsilon/2 \end{cases}$

$$\begin{matrix} a_0 & a_1 & a_2 & . & . & . & a_N & . & . & . & . \\ b_0 & b_1 & b_2 & . & . & . & . & . & b_M & . & . \end{matrix}$$

Choose M s.t. $b_0, b_1, \dots, b_M$ contains $a_0, a_1, \dots, a_N$ $\quad \{\sigma(1), \sigma(2), \dots, \sigma(N)\} \subseteq \{1,2,\dots,M\}$

$$m \ge M \Rightarrow \left|S - \sum_{k=0}^{m} b_k\right| = \left|S - \sum_{k=0}^{N} a_k\right| + \left|\sum_{\substack{subset\ outside \\ \{a_0,a_1,\dots,a_N\}}} b_k\right| \le \left|S - \sum_{k=0}^{N} a_k\right| + \sum_{k=N+1}^{\infty} |a_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore \sum_{k=0}^{\infty} b_k = S$

III.

Assume for simplicity that index starts at 1 and that there are only non-zero terms.

$\sum_{k=1}^{\infty}|a_k| = \infty$ and $|\sum_{k=1}^{\infty} a_k| < \infty$ with $a_k \in \mathbb{R}$.

Show $\forall S \in \mathbb{R} \exists \sigma: \mathbb{N}_1 \to \mathbb{N}_1$ (σ bijective i.e. a permutation) s.t. $\sum_{k=1}^{\infty} a_{\sigma(k)} = S$

Rearrange terms and you can get any sum you want.

Divide the series into two, one with negative terms replaced by zero $a_k^+ = \max(a_k, 0)$
and one with positive terms replaced by zero $a_k^- = \min(a_k, 0)$

$\sum_{k=1}^{\infty}|a_k|$ infinite and $\sum_{k=1}^{\infty} a_k$ finite $\Longrightarrow$ $\sum_{k=1}^{\infty} a_k^+ = \infty$ and $\sum_{k=1}^{\infty} a_k^- = -\infty$

Choose a sum S to attain

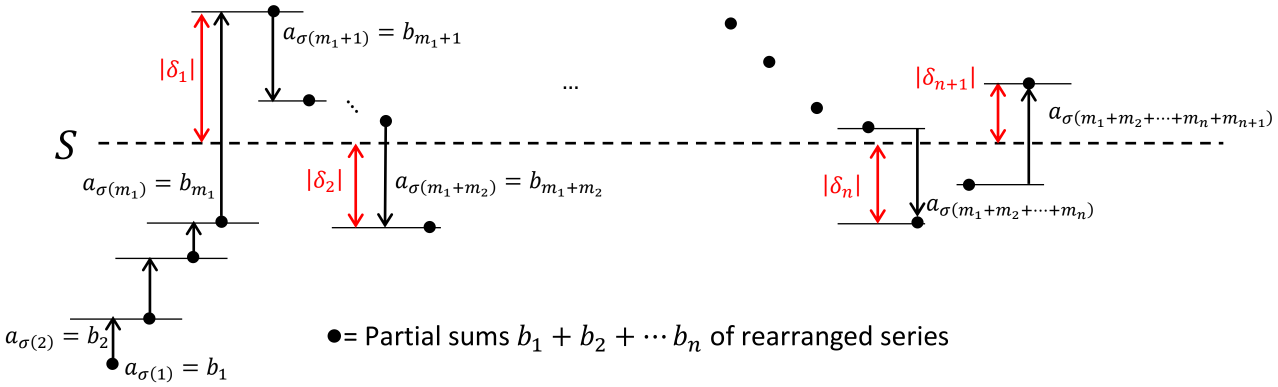
Pick non-zero terms from a_k^+ in order starting at first available term until the sum exceeds S.

$$\sum_{k=1}^{m_1-1} a_{\sigma(k)} \leq S < \sum_{k=1}^{m_1} a_{\sigma(k)}$$

Pick non-zero terms from a_k^- in order starting at first available term until the sum is less than S.

$$\sum_{k=1}^{m_1} a_{\sigma(k)} + \sum_{k=1}^{m_2} a_{\sigma(m_1+k)} < S \leq \sum_{k=1}^{m_1} a_{\sigma(k)} + \sum_{k=1}^{m_2-1} a_{\sigma(m_1+k)}$$

The process can be repeated indefinitely to get $\sigma(k)$ for every $k \in \mathbb{N}_1$.



The deviation of the partial sum from S in between changes of direction is limited by the deviation at the preceding change of direction and the deviation at change n, $|\delta_n|$ has an upper limit:

$$|\delta_n| = \left| S - \sum_{k=1}^{m_1+\cdots+m_n} b_k \right| < \left|a_{\sigma(m_1+\cdots+m_n)}\right|$$

$\left|\sum_{k=1}^{\infty} a_k\right| < \infty \Rightarrow |a_k| \to 0$ as $k \to \infty$ $\;\Rightarrow\;$ $\left|a_{\sigma(m_1+\cdots+m_n)}\right| \to 0$ as $n \to \infty$ since

$\sigma(m_1 + m_2 + \cdots + m_n) \to \infty$ as n $\to \infty$ by the sequential picking method

$\therefore \lim_{n\to\infty} |\delta_n| = 0$ and therefore $\lim_{n\to\infty} \sum_{k=1}^{n} a_{\sigma(k)} = S$

3.38 Show that every solution to $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_0 y = 0$
with characteristic polynomial $l(r) = \prod_{k=1}^{\nu}(r - r_k)^{n_k}$ is of the form
$y(x) = \sum_{k=1}^{\nu} P_k(x)e^{r_k x}$ with $\deg P_k < n_k$.

Proof by induction over the degree of $l(r)$.

$\deg l(r) = 1$
$y' + a_0 y = 0 \Rightarrow D(e^{a_0 x}y) = 0 \Rightarrow y = Ce^{-a_0 x}$ which is of the form $P_1(x)e^{r_1 x}$ with $\deg P_1 < 1$
$l(r) = r + a_0 \Rightarrow r_1 = -a_0$

Assume statement is true $\mathcal{L}(y) = 0$ with $\deg l(r) = k - 1$.

$\deg l(r) = k$
Let r_1 be a root of $l(r)$, $l(r) = l_1(r)(r - r_1)$ with $\deg l_1(r) = k - 1$.

Let $z = y' - r_1 y$

$l(D)y = 0 \Rightarrow l_1(D)(D - r_1)y = 0 \Rightarrow l_1(D)z = 0 \Rightarrow$ by induction assumption
$z(x) = \sum_{k=1}^{\nu} Q_k(x)\, e^{r_k x}$ with $\deg Q_k < n_k$ and $\deg Q_1 < n_1 - 1$
(If $n_1 = 1$, r_1 is a simple root and there will be no term with $k = 1$.)

$$y' - r_1 y = \sum_{k=1}^{\nu} Q_k(x)\, e^{r_k x} \Rightarrow D(e^{-r_1 x}y) = e^{-r_1 x}\sum_{k=1}^{\nu} Q_k(x)\, e^{r_k x} \Rightarrow$$

$$y(x) = e^{r_1 x}\left(\sum_{k=1}^{\nu}\left(\int_0^x Q_k(t)e^{(r_k - r_1)t}dt\right) + C\right)$$

$$k = 1\colon e^{r_1 x}\left(\int_0^x Q_1(t)dt + C_1\right) = P_1(x)e^{r_1 x} \text{ with } \deg P_1 = 1 + \deg Q_1 < n_1$$

$$k > 1\colon e^{r_1 x}\left(\int_0^x Q_k(t)e^{(r_k - r_1)t}dt + C\right) = \begin{matrix}\text{after repeated}\\ \text{integration}\\ \text{by parts}\end{matrix} = P_k(x)e^{r_k x} \text{ with } \deg P_k < n_k$$

$\therefore$ Statement true for $n = k$ and by induction true for $\mathcal{L}(y) = 0$ of every degree.

3.39 Homogeneous linear recurrence relation with constant coefficients of order n:

$$a_k = c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_n a_{k-n} \ (*) \quad a_i, c_i, r_i \in \mathbb{C}$$

with characteristic polynomial $p(t) = t^n - \sum_{i=1}^{n} c_i t^{n-i} = \prod_{j=1}^{\nu}(t - r_j)^{n_j}$, $n_1 + \cdots + n_\nu = n$

Show that $a_k = \sum_{j=1}^{\nu} P_j(k) {r_j}^k$ with polynomials P_j of degree less than n_j solves $(*)$.

With sequence $\langle a_k \rangle \equiv \langle a_0, a_1, a_2, \dots \rangle$ and linear operator $\mathcal{L}\langle a_k \rangle = \langle a_k - \sum_{i=1}^{n} c_i a_{k-i} \rangle$
it is enough to show that the sequence $\langle a_k \rangle = \langle P_j(k) {r_j}^k \rangle$ with $\deg P_j < n_j$ solves $\mathcal{L}\langle a_k \rangle = 0$.

Forward shift operator	$E\langle x_k \rangle \equiv \langle x_{k+1} \rangle$
Forward difference operator	$\Delta \equiv E - I$
	$\Delta\langle x_k \rangle = \langle x_{k+1} - x_k \rangle$
Forward difference operator	$\Delta_h[f](x) \equiv f(x+h) - f(x)$
acting on a function f	$\Delta[f](x) \equiv f(x+1) - f(x)$
For polynomial P of deg P=n	${\Delta_h}^k[P](x) = 0$ if $k > n$
	$\deg({\Delta_h}^k[P]) = n - k$ (Proved by induction)

Show $a_k - \sum_{i=1}^{n} c_i a_{k-i} = 0$ for every $k > n$

$$E^k a_0 - \sum_{i=1}^{n} c_i E^{k-i} a_0 = \prod_{j=1}^{\nu}(E - r_j)^{n_j} E^{k-n} a_0 = \prod_{j=1}^{\nu}(E - r_j)^{n_j} a_{k-n}$$

Let $a_k = P(k){r_j}^k$ with a polynomial P of $\deg P < n_j$

$$(E - r_j)^{n_j} P(m) r_j^m = \sum_{i=0}^{n_j} \binom{n_j}{i} E^i (-r_j)^{n_j - i} P(m) r_j^m = \sum_{i=0}^{n_j} \binom{n_j}{i} (-r_j)^{n_j - i} P(m+i) {r_j}^{m+i} =$$

$$(-1)^{n_j} {r_j}^{n_j + m} \sum_{i=0}^{n_j} \binom{n_j}{i} (-1)^i P(m+i) \begin{bmatrix} Q(n_j - i) \equiv P(m+i) \\ Q \text{ is a polynomial} \\ \deg Q = \deg P \end{bmatrix} = c \cdot \sum_{i=0}^{n_j} \binom{n_j}{i} (-1)^i E^{n_j - i} Q(0) =$$

$c \cdot \Delta^{n_j} Q(0) = 0$ since $\deg Q = \deg P < n_j$

$\therefore a_k - \sum_{i=1}^{n} c_i a_{k-i} = 0$ for every $k > n$ when a_k is of the form $\sum_{j=1}^{\nu} P_j(k) {r_j}^k$ with $\deg P_j < n_j$.

Discretized versions of PDEs corresponds to multidiensional recurrence relations.
An example of such a relation is Pascal's triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

with boundary condition $\binom{n}{0} = \binom{n}{n} = 1$.

Linear homogeneous ODEs with constant coefficients are solved with ansatz $f(x) = e^{rx}$.
The discretized version is solved with ansatz $a_k = r^k$, in both cases with r a solution to a characteristic polynomial. A Taylor series for the solution to the ODE gives a connection.

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \text{ with } c_k = f^k(0)/k!$$

The ODE leads to a recurrence relation for the coefficients c_k.

$$ay'' + by' + cy = 0 \to af^{(k)}(0) + bf^{(k+1)}(0) + cf^{(k+2)}(0) = 0$$
$$x_k = f^{(k)}(0) \to ax_k + bx_{k+1} + cx_{k+2} = 0 \quad , \quad f(x) = e^{rx} \to f^{(k)}(0) = r^k$$

3.40 The weighted power mean M_p of $x_1, \dots, x_n \in \mathbb{R}^+$
with weights $w_i \in \mathbb{R}^+$ and $\sum_{i=1}^n w_i = 1$ is defined by

$M_p(x_1, \dots, x_n) = (\sum_{i=1}^n w_i {x_i}^p)^{1/p}$ for $p \in \mathbb{R} \setminus \{0\}$
$M_0(x_1, \dots, x_n) = \prod_{i=1}^n {x_i}^{w_i}$
$M_{-\infty}(x_1, \dots, x_n) = \min(x_1, \dots, x_n)$
$M_{\infty}(x_1, \dots, x_n) = \max(x_1, \dots, x_n)$

$p = -1$: harmonic mean
$p = 0$: geometric mean
$p = 1$: arithmetic mean
$p = 2$: square mean

Show:
$\lim_{p \to 0} M_p = M_0$
$\lim_{p \to -\infty} M_p = M_{-\infty}$
$\lim_{p \to \infty} M_p = M_{\infty}$
$p < q \implies M_p(x_1, \dots, x_n) \le M_q(x_1, \dots, x_n)$ with equality iff $x_1 = x_2 = \dots = x_n$
($\min \le H.M \le G.M \le A.M \le S.M \le \max$)

4.1 Prove $p^2|(2^{p(p-1)}-1)$ when p is a prime.

Fermat's Little theorem gives $2^{p-1} \equiv 1 \pmod{p}$, $2^{p-1} = kp+1$ for some $k \in \mathbb{Z}$.

$2^{p(p-1)} - 1 = (kp+1)^p - 1 = p^2(\dots)$ by the binomial theorem

$\therefore p^2|2^{p(p-1)} - 1$

This can illustrates the ABC conjecture about numbers (a, b, c) such that $a + b = c$:

> For any $\varepsilon > 0$ there is only finitely many triples (a, b, c) with positive, coprime integers s.t.
>
> $$\mathrm{rad}(\mathrm{abc})^{1+\varepsilon} < c$$

The condition $\varepsilon > 0$ can not be relaxed, the number of triples with $\mathrm{rad}(abc) < c$ is infinite, it's even true that the number of triples with $\mathrm{rad}(abc) < \alpha c$ is infinite for any $\alpha > 0$.

The function $\mathrm{rad}(n)\colon \mathbb{Z}^+ \to \mathbb{Z}^+$is given by $\mathrm{rad}(\prod {p_i}^{n_i}) = \prod p_i$

$a = 1\ \ b = 2^{p(p-1)n} - 1\ \ c = 2^{p(p-1)}$ with p a prime and $n \in \mathbb{Z}^+$ satisfies $a + b = c$.

$$b = \left(2^{p(p-1)}\right)^n - 1^n = \left(2^{p(p-1)} - 1\right)(\dots) = p^2(\dots) \Rightarrow b = p^2 \cdot \frac{b}{p^2} \quad \text{with } \frac{b}{p^2} \text{ an integer}$$

$$\mathrm{rad}(abc) = 2\mathrm{rad}(b) = 2p \cdot \mathrm{rad}\left(\frac{b}{p^2}\right) \le 2p\frac{b}{p^2} = \frac{2b}{p} < \frac{2c}{p}$$

For any prime $p > 2/\alpha$ and $n \in \mathbb{Z}^+$: $\mathrm{rad}(abc) < \alpha c$

4.2 Show that if a fraction a/p with $0 < a < p$ and p a prime has a decimal expansion with even period $a/p = 0.\overline{r_1 \dots r_n r_{n+1} \dots r_{2n}}$ then $r_i + r_{i+n} = 9$

Example: $\frac{1}{17} = 0.\overline{\underbrace{05882352}_{A}\,\underbrace{94117647}_{B}}$ $\quad \begin{array}{r} 05882352 \\ \underline{94117647} \\ 99999999 \end{array} \quad A + B = 10^n - 1$

$$\frac{a}{p} = 0.\overline{r_1 \dots r_{2n}} \Rightarrow \frac{a}{p} \cdot 10^{2n} = N + \frac{a}{p} \Rightarrow (10^{2n} - 1)\frac{a}{p} = N \quad (N = r_1 \dots r_{2n} = A \cdot 10^n + B)$$

$$\frac{a}{p} = \frac{N}{10^{2n} - 1}$$

$p|10^{2n} - 1$ since $\frac{(10^{2n}-1)a}{p} \in \mathbb{Z}$ and $0 < a < p$

$10^k - 1$ is not a multiple of p for any $k < 2n$ since that would make the period of a/p less than $2n$.

$$\frac{a}{p} = \frac{N}{(10^n + 1)(10^n - 1)} \quad \left(\begin{matrix} 10^{2n} - 1 \text{ is a multiple of } p \\ 10^n - 1 \text{ is not a multiple of } p \end{matrix} \Rightarrow 10^n + 1 \text{ is a multiple of } p \right)$$

$$\frac{a(10^n + 1)}{p} = \frac{N}{10^n - 1} \in \mathbb{Z} \Rightarrow \begin{matrix} N \text{ is a multiple of } 10^n - 1 \\ N \equiv 0 \pmod{10^n - 1} \end{matrix}$$

$$\left.\begin{matrix} N = A \cdot 10^n + B \\ 10^n \equiv 1 \pmod{10^n - 1} \end{matrix} \Rightarrow N \equiv A + B \pmod{10^n - 1} \Rightarrow A + B \equiv 0 \pmod{10^n - 1} \atop \begin{matrix} 0 < A < 10^n - 1 \\ 0 < B < 10^n - 1 \end{matrix} \Rightarrow 0 < A + B < 2(10^n - 1) \right\} \Rightarrow A + B = 10^n - 1$$

4.3. There are infinitely many triples of positive integers (a, b, c) with $\gcd(a, b, c) = 1$ s.t. $a + b = c$ and $q(a, b, c) > 1$ where:
$q(a, b, c) = \log(c) / \log\big(\mathrm{rad}(abc)\big)$, $\mathrm{rad}\big(\prod p_i^{k_i}\big) = \prod p_i$.
The abc-conjecture states: $\varepsilon > 0 \Rightarrow$ only finitely many triples has $q(a, b, c) > 1 + \varepsilon$.
If the abc-conjecture is true then there is a maximal value of $q(a, b, c)$.
Assume the abc-conjecture and that $q(a, b, c)$ is always less than 2.
Show that Fermat's last theorem $a^n + b^n = c^n$ with $\gcd(a, b, c) = 1$ holds for $n \geq 6$.
(The abc-conjecture says nothing about the limit of $q(a, b, c)$, biggest known case is 1.63.)

Use the assumption on a counter example of Fermat's last theorem $\underbrace{a^n}_{A} + \underbrace{b^n}_{B} = \underbrace{c^n}_{C}$ where $n \geq 6$.

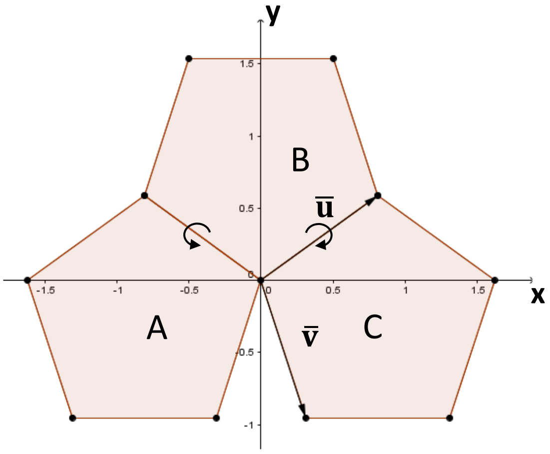
a, b, c are assumed co-prime.

The quality $q(A, B, C)$ for the counter example gives $q(A, B, C) = \log(c^n) / \log\big(\mathrm{rad}(a^n b^n c^n)\big)$

$q(A, B, C) < 2 \implies \log(c^n) < 2\log\big(rad(a^n b^n c^n)\big) \implies$
$c^n < \big(\mathrm{rad}(a^n b^n c^n)\big)^2 = \mathrm{rad}(abc)^2 \leq (abc)^2 < (c^3)^2 = c^6 \implies$
$n < 6$ is a contradiction

$\therefore$ There can be no counterexample to FLT for $n \geq 6$ if we assume ABC and $q(A, B, C) < 2$.

5.x Show that the inner angle between two adjoining planes in a regular dodecahedron equals $\nu = 2\arctan(\varphi)$ where $\varphi \equiv \frac{\sqrt{5}+1}{2}$ is the golden ratio. Solve it by using a matrix for rotation.



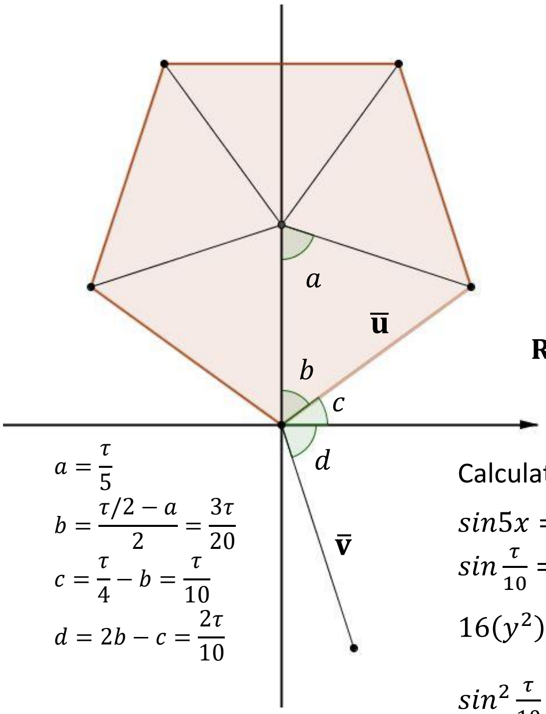
Rotate the pentagons A and C simultaneously until they meet, i.e. rotate vector $\bar{\mathbf{v}}$ around unit vector $\bar{\mathbf{u}}$ until it points backwards i.e. $\mathbf{R}\bar{\mathbf{v}} \cdot \bar{\mathbf{e}}_x = 0$.

The rotation matrix **R** around $\bar{\mathbf{u}} = (\mathrm{u}_x, \mathrm{u}_y, \mathrm{u}_z)$ with angle ω is:

$$\begin{pmatrix} \mathrm{u}_x^2(1-c)+c & u_x u_y(1-c)-u_z s & u_z u_x(1-c)+u_y s \\ u_y u_x(1-c)+u_z s & \mathrm{u}_y^2(1-c)+c & u_y u_z(1-c)-u_x s \\ u_z u_x(1-c)-u_y s & u_z u_y(1-c)+u_x s & \mathrm{u}_z^2(1-c)+c \end{pmatrix}$$

where $c = \cos\omega$ and $s = \sin\omega$. Calculate angles in unit $\tau = 1$ turn.

$\bar{\mathbf{u}} = (\cos\alpha, \sin\alpha, 0)$ and $\bar{\mathbf{v}} = (\cos 2\alpha, -\sin 2\alpha, 0)$ where $\alpha = \tau/10$



$$a = \frac{\tau}{5}$$
$$b = \frac{\tau/2 - a}{2} = \frac{3\tau}{20}$$
$$c = \frac{\tau}{4} - b = \frac{\tau}{10}$$
$$d = 2b - c = \frac{2\tau}{10}$$

$$\begin{aligned} \mathbf{R}\bar{\mathbf{v}} \cdot \bar{\mathbf{e}}_x &= \cos 2\alpha[\mathrm{u}_x^2(1-c)+c] - \sin 2\alpha[u_x u_y(1-c)] \\ &= \cos 2\alpha[c u_y^2 + u_x^2] - \sin 2\alpha[-c u_x u_y + u_x u_y] \\ &= \cos\omega \cdot \sin\alpha[\cos 2\alpha \cdot \sin\alpha + \sin 2\alpha \cdot \cos\alpha] + \\ &\quad \cos\alpha[\cos 2\alpha \cdot \cos\alpha - \sin 2\alpha \cdot \sin\alpha] \end{aligned}$$

$$\mathbf{R}\bar{\mathbf{v}} \cdot \bar{\mathbf{e}}_x = 0 \;\Rightarrow \cos\omega = \frac{\cos\alpha[\sin 2\alpha \cdot \sin\alpha - \cos 2\alpha \cdot \cos\alpha]}{\sin\alpha[\cos 2\alpha \cdot \sin\alpha + \sin 2\alpha \cdot \cos\alpha]} = \frac{\cos^2\alpha(3\sin^2\alpha - \cos^2\alpha)}{\sin^2\alpha(3\cos^2\alpha - \sin^2\alpha)}$$

Calculating $\sin\alpha$ for $\alpha = \frac{\tau}{10}$:

$$\sin 5x = Im[(\cos x + i \cdot \sin x)^5] = 16\sin^5 x - 20\sin^3 x + 5\sin x$$

$$\sin\frac{\tau}{10} = y \text{ ger } 0 = 16y^5 - 20y^3 + 5y$$

$$16(y^2)^2 - 20y^2 + 5 = 0 \text{ ger } y^2 = \frac{5 \pm \sqrt{5}}{8}$$

$$\sin^2\frac{\tau}{10} < \sin^2\frac{\tau}{8} = \frac{1}{2} \;\Longrightarrow\; y^2 = \frac{5-\sqrt{5}}{8} \;\Longrightarrow \sin\alpha = \sqrt{\frac{5-\sqrt{5}}{8}} \Longrightarrow \cos\alpha = \frac{1}{4}(1+\sqrt{5})$$

$$\left.\begin{aligned} \cos^2\alpha(3\sin^2\alpha - \cos^2\alpha) &= 1/4 \\ \sin^2\alpha(3\cos^2\alpha - \sin^2\alpha) &= \sqrt{5}/4 \end{aligned}\right. \;\Longrightarrow\; \cos\omega = 1/\sqrt{5} \;\;\Longrightarrow\;\; \nu = \frac{\tau}{2} - \arccos\frac{1}{\sqrt{5}}$$

Remains to show: $\frac{\tau}{2} - \arccos\frac{1}{\sqrt{5}} = 2\arctan\varphi \quad \varphi = \frac{\sqrt{5}+1}{2}$

$$\cos\left(\pi - \arccos\frac{1}{\sqrt{5}}\right) = -\cos\left(\arccos\frac{1}{\sqrt{5}}\right) = -\frac{1}{\sqrt{5}}$$

$$\begin{aligned} \cos(2\arctan\varphi) &= \cos^2(\arctan\varphi) - \sin^2(\arctan\varphi) \\ &= \frac{1}{1+\varphi^2} - \frac{\varphi^2}{1+\varphi^2} \\ &= \frac{1-(6+2\sqrt{5})/4}{1+(6+2\sqrt{5})/4} \\ &= -\frac{1}{\sqrt{5}} \end{aligned}$$

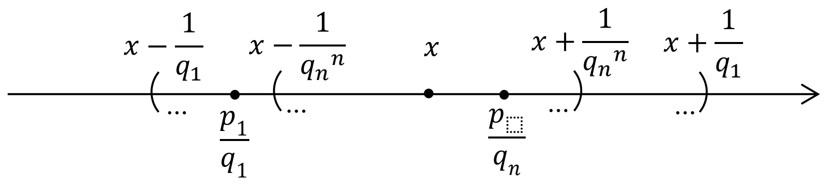
$$\boxed{\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \frac{1}{\tan^2 x} + 1 &= \frac{1}{1-\cos^2 x} \\ \cos x &= \frac{1}{\sqrt{1+\tan^2 x}} \end{aligned}}$$

$$\boxed{\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \frac{1}{1-\sin^2 x} \\ \sin x &= \frac{\tan x}{\sqrt{1+an^2 x}} \end{aligned}}$$

Conclusion: The inner angle between the planes in a regular dodecahedron is $2\arctan\frac{\sqrt{5}+1}{2} \approx 116{,}57°$

C.1 Show that the Liouville numbers $\mathbb{L}$ are transcendental and that they form an uncountable dense subset of $\mathbb{R}$ with Lebesgue measure zero.

$$\mathbb{L} = \left\{x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N}_1 \; \exists (p,q) \in \mathbb{Z} \times \mathbb{N}_2 \left(\left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right) \right\} \qquad \mathbb{N}_k = \{k, k+1, \dots\}$$



(I) Liouville numbers ($\mathbb{L}$) are transcendental

Theorem 7 from appendix C (Liouville's theorem)

$$\left[\forall C \in \mathbb{R}^+ \; \forall n \in \mathbb{N}_1 \; \exists (p,q) \in \mathbb{Z} \times \mathbb{N}_1 \; \left(0 < \left| x - \frac{p}{q} \right| \le \frac{C}{q^n} \right) \right] \Rightarrow x \in \mathbb{R} \setminus \mathbb{A}$$

Show

$$\left[\; \forall n \in \mathbb{N}_1 \; \exists (p,q) \in \mathbb{Z} \times \mathbb{N}_2 \; \left(0 < \left| x - \frac{p}{q} \right| \le \frac{1}{q^n} \right) \right] \Rightarrow x \in \mathbb{R} \setminus \mathbb{A}$$

Choose $r \in \mathbb{N}_1$ s.t. $\frac{1}{2^r} \le C$ and let $m = r + n$

$$x \in \mathbb{L} \;\; \Rightarrow \; \exists p, q > 1 \text{ s.t. } \; 0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^m} = \frac{1}{q^{r+n}} \le \frac{1}{2^r q^n} \le \frac{C}{q^n} \quad \Rightarrow \quad x \in \mathbb{R} \setminus \mathbb{A}$$

(II) $\mathbb{L}$ is an uncountable set

If $(a_1, a_2, \dots)$ is a sequence with $a_k \in \{0,1,\dots,b-1\}$ $(b \in \mathbb{N}_2)$ and infinitely many $a_k \neq 0$ then

$$x = \sum_{k=1}^{\infty} \frac{a_k}{b^{k!}} = (0.a_1 a_2 000 a_3 00000000000000000 a_4 00 \dots)_b \text{ is not periodic, so it's not rational.}$$

For $n \in \mathbb{N}_1$ choose $q_n = b^{n!}$ and $p_n = q_n \sum_{k=1}^{n} \frac{a_k}{b^{k!}}$ then:

$$0 < \left| x - \frac{p_n}{q_n} \right| = \sum_{k=n+1}^{\infty} \frac{a_k}{b^{k!}} \le \sum_{k=(n+1)!}^{\infty} \frac{b-1}{b^k} = \frac{b-1}{b^{(n+1)!}} \sum_{k=0}^{\infty} \frac{1}{b^k} = \frac{b}{b^{(n+1)!}} \le \frac{b^{n!}}{b^{(n+1)!}} = \frac{1}{b^{n \cdot n!}} = \frac{1}{q_n{}^n}$$

$n! - (n+1)! = n! - (n \cdot n! + n!) = -n \cdot n!$ x is a Liouville number

So there are at least as many numbers in $\mathbb{L}$ as there are sequences $(c_1, c_2, \dots)$ with $c_k \in \{1,2\}$ which are in one-to- one correspondence with reals in $[0,1] = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{2^k} : a_k \in \{0,1\} \right\}$
The cardinality of $\mathbb{L}$ must equal $\mathbb{R}$, there are uncountably many Liouville numbers.

(III) $\mathbb{L}$ is a dense set in $\mathbb{R}$

For any open interval $J_k = (x - \delta_k, x + \delta_k)$ centered on $x \in \mathbb{R}$ we can pick $p/q \in \mathbb{Q}$ in J_k and $\alpha_k = p/q + 2^{-n} \sum_{i=1}^{\infty} 2^{-i!}$ will belong to both $\mathbb{L}$ and J_k for n large enough.
With $\delta_k = 1/k$ we get a sequence α_k in $\mathbb{L}$ with x as a limit point so $\mathbb{L}$ is a dense set in $\mathbb{R}$.

(IV) $\mathbb{L}$ is a set with Lebesgue measure zero.

For $n \in \mathbb{N}_3$ and $q \in \mathbb{N}_2$ set:

$$V_{n,q} = \bigcup_{p=-\infty}^{\infty} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right) \qquad \mathbb{L} \subseteq \bigcup_{q=2}^{\infty} V_{n,q} \qquad V_{3,q} \supseteq V_{4,q} \supseteq \cdots$$

At the same time:

$$\mathbb{L} \cap (-m,m) \subseteq \bigcup_{q=2}^{\infty} V_{n,q} \cap (-m,m) \subseteq \bigcup_{q=2}^{\infty} \bigcup_{p=-mq}^{mq} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right)$$

$$\left|\left(\frac{p}{q} + \frac{1}{q^n}\right) - \left(\frac{p}{q} - \frac{1}{q^n}\right)\right| = \frac{2}{q^n} \quad \text{and} \quad n > 2 \quad \Longrightarrow$$

$$\lambda\big(\mathbb{L} \cap (-m,m)\big) \le \sum_{q=2}^{\infty} \sum_{p=-mq}^{mq} \frac{2}{q^n} = \sum_{q=2}^{\infty} \frac{2(2mq+1)}{q^n} \le (4m+1) \sum_{q=2}^{\infty} \frac{1}{q^{n-1}} \le (4m+1) \int_1^{\infty} \frac{dq}{q^{n-1}} \le \frac{4m+1}{n-2}$$

$$\lim_{n\to\infty} \frac{4m+1}{n-2} = 0 \quad \Longrightarrow$$

For every positive integer m, $\mathbb{L} \cap (-m,m)$ has Lebesgue measure zero,
which means that the Lebesgue measure of $\mathbb{L}$ is zero.
The transcendental numbers $\mathbb{R} \setminus \mathbb{A}$ are the complement of a null set $\mathbb{A}$ so
$\lambda(\mathbb{L}) = 0$ and $\lambda(\mathbb{R} \setminus \mathbb{A}) = \infty$. ■

C.2 Show that the Bernoulli numbers satisfy $B_{2k+1} = 0$ for $k \ge 1$.

$$\langle B_n \rangle = \langle 1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \dots \rangle$$

The Bernoulli numbers can be defined as the coefficients of a power series $\dfrac{z}{e^z - 1} = \sum_{n \ge 0} B_n \dfrac{z^n}{n!}$

Add $z/2$ to cancel the term $B_1 z/1! = -z/2$.

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} \cdot \frac{e^z + 1}{e^z - 1} = \frac{z}{2} \cdot \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} \qquad (= \frac{z}{2} \coth \frac{z}{2})$$

This function is an even function $f(z) = f(-z)$ so all its odd powers of z must be zero.
In the expansion of $z/(e^z - 1)$ all coefficients of z^n with odd n larger than one must be zero.
$B_3 = B_5 = B_7 = \cdots = 0$.

Furthermore:

$$z \coth z = \frac{2z}{e^{2z} - 1} + \frac{2z}{2} = \sum_{n\ge0} B_{2n} \frac{(2z)^{2n}}{(2n)!} = \sum_{n\ge0} 4^n B_{2n} \frac{z^{2n}}{(2n)!}$$

$$\begin{array}{l} \sin z = -i \sinh z \\ \cos z = \cosh iz \end{array} \to \cot z = i \coth iz \to \quad z \cot z = \sum_{n\ge0} (-4)^n B_{2n} \frac{z^{2n}}{(2n)!}$$

C.3 Prove that ordinary and binomial convolutions,

$$\langle f_n\rangle \star \langle g_n\rangle = \langle \sum_{k=0}^{n} f_k g_{n-k}\rangle \text{ and } \langle f_n\rangle \star^b \langle g_n\rangle = \langle \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}\rangle$$

are commutative and associative operators with identity $\langle 1,0,0,\dots\rangle$ and have a unique inverse for sequences $\langle a_0, a_1, a_2, \dots\rangle$ with $a_0 \neq 0$.

Ordinary convolution **Binomial convolution**

Elements in sequences are assumed to belong to a field, so properties of $\mathbb{R}$ are assumed.

Commutativity

$\langle f_n\rangle \star \langle g_n\rangle = \langle \sum_{k=0}^{n} f_k g_{n-k}\rangle$

Sum in opposite order $k \curvearrowright n-k$ gives

$\langle f_n\rangle \star \langle g_n\rangle = \langle g_n\rangle \star \langle f_n\rangle$

$\langle f_n\rangle \star^b \langle g_n\rangle = \langle \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}\rangle$

Sum in opposite order and $\binom{n}{k} = \binom{n}{n-k}$ gives

$\langle f_n\rangle \star^b \langle g_n\rangle = \langle g_n\rangle \star^b \langle f_n\rangle$

Associativity

$(\langle f_n\rangle \star \langle g_n\rangle) \star \langle h_n\rangle =$

$$\sum_{k_i} \underbrace{f_{k_1} g_{k_2} h_{k_3}}_{k_1+k_2+k_3=n} =$$

$\langle f_n\rangle \star (\langle g_n\rangle \star \langle h_n\rangle)$

$(\langle f_n\rangle \star^b \langle g_n\rangle) \star^b \langle h_n\rangle =$

$$\sum_{m+k_3=n} \binom{n}{m, k_3} \left[\sum_{k_1+k_2=m} \binom{m}{k_1, k_2} f_{k_1} g_{k_2}\right] h_{k_3}$$

$$\sum_{k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} f_{k_1} g_{k_2} h_{k_3} = \langle f_n\rangle \star^b (\langle g_n\rangle \star^b \langle h_n\rangle)$$

Identity

The sequence $\langle 1,0,0,\dots\rangle = \langle \delta_{n0}\rangle$ is the identity element for both types of convolution

$\langle f_n\rangle \star \langle \delta_{n0}\rangle = \langle \sum_{k=0}^{n} f_k \delta_{n-k0}\rangle = \langle f_n\rangle$ $\qquad$ $\langle f_n\rangle \star^b \langle \delta_{n0}\rangle = \langle \sum_{k=0}^{n} \binom{n}{k} f_k \delta_{n-k0}\rangle = \langle f_n\rangle$

Inverse

Let $\langle f_n\rangle$ be a sequence with $f_0 \neq 0$

$\langle f_n\rangle \star \langle g_n\rangle = \langle 1,0,0,\dots\rangle$

$n = 0$:

$f_0 g_0 = 1 \leftrightarrow g_0 = f_0^{-1}$

Assume g_k is uniquely defined for each $k < n$.

$k = n$:

$f_0 g_n + \sum_{k=1}^{n} f_k g_{n-k} = 0 \leftrightarrow g_n = f_0^{-1}(-\sum_{k=1}^{n} f_k g_{n-k})$ Each g_i needed is uniquely defined

All elements in the inverse of $\langle f_n\rangle$ is uniquely defined as long as $f_0 \neq 0$.

Existence of a unique inverse of $\langle f_n\rangle$ with $f_0 \neq 0$ for binomial convolution is proved similarly.

C.4 Use the formula for the resultant, $\Delta(P) = (-1)^{n(n-1)/2} R(P, P')/a_n$ of $P = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n(z - r_1)(z - r_2) \ldots (z - r_n)$ with $R(P, Q) = |S_{P,Q}|$ where $S_{P,Q}$ is the Sylvester matrix to find the discriminant of $ax^4 + bx^3 + cx^2 + dx + e$ and check that the result is in accordance with the definition $\Delta(P) \equiv a_n^{2n-2} \cdot \prod_{1 \le i < j \le n} (r_i - r_j)^2$.

With Mathematica to carry out the calculations:

```
In[1]:= n =.; m =.; s =.;
In[2]:= n = 4; m = n - 1; s = {};
In[3]:= p = {a, b, c, d, e, 0, 0};
In[4]:= For[i = 0, i < m, i++, s = Append[s, RotateRight[p, i]]]
In[5]:= q = {4 a, 3 b, 2 c, d, 0, 0, 0};
In[6]:= For[i = 0, i < n, i++, s = Append[s, RotateRight[q, i]]]
In[7]:= MatrixForm[s]   (*Sylvester Matrix*)
Out[7]//MatrixForm=
```

$$\begin{pmatrix} a & b & c & d & e & 0 & 0 \\ 0 & a & b & c & d & e & 0 \\ 0 & 0 & a & b & c & d & e \\ 4a & 3b & 2c & d & 0 & 0 & 0 \\ 0 & 4a & 3b & 2c & d & 0 & 0 \\ 0 & 0 & 4a & 3b & 2c & d & 0 \\ 0 & 0 & 0 & 4a & 3b & 2c & d \end{pmatrix}$$

```
In[8]:= discrS = Expand[(-1)^(n*(n-1)/2)*Det[s]/a]  (*Discriminant with S.M.*)
```

Out[8]= $b^2 c^2 d^2 - 4 a c^3 d^2 - 4 b^3 d^3 + 18 a b c d^3 - 27 a^2 d^4 - 4 b^2 c^3 e + 16 a c^4 e + 18 b^3 c d e - 80 a b c^2 d e - 6 a b^2 d^2 e + 144 a^2 c d^2 e - 27 b^4 e^2 + 144 a b^2 c e^2 - 128 a^2 c^2 e^2 - 192 a^2 b d e^2 + 256 a^3 e^3$

```
In[9]:= discrS - Discriminant[a x^4 + b x^3 + c x^2 + d x + e, x]  (*Check with Mathematica function*)
Out[9]= 0
In[10]:= v = Product[r[i] - r[j], {i, 1, 4}, {j, i + 1, 4}];  (*Vandermonde determinant*)
In[11]:= discrDefR = a^(2*n - 2)*v^2;  (*Discriminant as defined by root function*)
In[12]:= br = -a*Sum[r[i], {i, 1, 4}];  (* coeffients by roots, Vieta's formulas *)
In[13]:= cr = a*Sum[r[i]*r[j], {i, 1, 4}, {j, i + 1, 4}];
In[14]:= dr = -a*Sum[r[i]*r[j]*r[k], {i, 1, 4}, {j, i + 1, 4}, {k, j + 1, 4}];
In[15]:= er = a*Product[r[i], {i, 1, 4}]; (* Discriminant from Sylvester matrix, with roots *)
In[16]:= discrSR = discrS /. {b → br, c → cr, d → dr, e → er};
In[17]:= Simplify[discrSR - discrDefR]  (* Check discriminant from S.M. same as deinition *)
Out[17]= 0
```

Discriminant with terms in lexical order.

$\Delta(ax^4 + bx^3 + cx^2 + dx^2 + e) =$

$256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e$

$+80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2$

C.5 The resultant $R(f,g)$ of two polynomials with coefficients in a field $\mathbb{F}$
where $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$, $(a_n \neq 0, b_m \neq 0)$
with roots $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ in the algebraic closure of $\mathbb{F}$
can be defined in two alternate ways:

1. $R_1(f,g) \equiv a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j)$

$$\mathbf{2.}\quad R_2(f,g) \equiv \underbrace{\begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & \cdots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & \cdots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & b_2 & b_1 & b_0 \end{vmatrix}}_{m+n \text{ columns}} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} m \text{ rows} \\ \left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} n \text{ rows} \end{array} = |S_{n,m}|$$

Show that the two definitions are equivalent.

For convenience in the proof, extend the definitions to cases where polynomials f and g may start with initial zero coefficients. Notation $R_{n,m}(f,g)$ is used to show that initial zeros have been added in the front of f or g when $\deg(f) < n$ or $\deg(g) < m$.

The proof will be by induction over $n + m$.

By definition $\prod_{k \in \emptyset} c_k = 1$, applied to the product of eigenvalues it gives $|M| = 1$ for a 0×0 matrix.
$n = 0$ and $m = 0$ gives $R_1(f,g) = 1$ and $R_2(f,g) = 1$
$n = 0$ gives $R_1(f,g) = a_0^m$ and $S_{0,m} = a_0 I_m \to R_2(f,g) = a_0^m$
$m = 0$ gives $R_2(f,g) = b_0^n$ and $S_{n,0} = b_0 I_n \to R_2(f,g) = b_0^n$

Lemma 1.

$$R_1(f,g) \equiv a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j) = \begin{cases} a_n^m \prod_{i=1}^{n} g(\alpha_i) & (1) \\ (-1)^{nm} b_m^n \prod_{j=1}^{m} f(\beta_j) & (2) \end{cases}$$

These statements follows from $g(x) = b_m \prod_{j=1}^{m} (x - \beta_j)$ and $f(x) = a_n \prod_{i=1}^{n} (x - \alpha_i)$ and there are nm factors $\beta_j - \alpha_i$ in (2) of the wrong sign. In the same way there are mn row permutations needed to switch places between upper and lower blocks in $S_{n,m}$. $R_{m,n}(g,f) = (-1)^{nm} R_{n,m}(f,g)$.

Lemma 2.

If $\deg(g) \le k \le m$ then $R_{n,m}(f,g) = a_n^{m-k} R_{n,k}(f,g)$ (1)
If $\deg(f) \le k \le n$ then $R_{n,m}(f,g) = (-1)^{(n-k)m} b_m^{n-k} R_{k,m}(f,g)$ (2)
(1): first column in $S_{n,m}$ is 0 except for a_n. Expanding the determinant repeatedly gives:
$R_{n,m}(f,g) = a_n R_{n,m-1}(f,g) = \cdots = a_n^{m-k} R_{n,k}(f,g)$
(2) follows similarly by first using $R_{m,n}(g,f) = (-1)^{nm} R_{n,m}(f,g)$.

Lemma 3.
Let f, g and h be polynomials with $\deg(f) \leq n$ and $\deg(g) \leq m$.
$n \geq m$ and $\deg(h) \leq n - m \Rightarrow R_{n,m}(f + hg, g) = R_{n,m}(f, g)$ (1)
$n \leq m$ and $\deg(\mathrm{h}) \leq m - n \Rightarrow R_{n,m}(f, g + hf) = R_{n,m}(f, g)$ (2)

(1): $\deg(f + hg) \leq \max(\deg(f), \deg(hg)) \leq \max(n, n - m + m) = n$.
$S_{n,m}(f + hg)$ is obtained from $S_{n,m}(f, g)$ by row operations
that do not change its determinant $R_{n,m}(f, g)$. If $h(x) = c_l x^l + \cdots + c_0$
add c_k times row $n + i - k$ to row i for $i = 1, \ldots, m$ and $k = 0, \ldots, l$.
(2) is shown in the same manner or by using $R_{m,n}(g, f) = (-1)^{nm} R_{n,m}(f, g)$.

Use the induction assumption to assume $R_2(f, g)$ equals the formulas (1) and (2)
of lemma 1 for all smaller values of $n + m$.

Case 1: $0 < n \leq m$ with $\deg(f) = n$ and $\deg(g) = m$.
$g/f \rightarrow \ g = qf + r$ with $\deg(r) < \deg(f)$ and $\deg(q) = m - n$. Lemma 3 gives

$$R_{n,m}(f, g) = R_{n,m}(f, g - qf) = R_{n,m}(f, r).$$

Case 1a: If $r \neq 0$, let $k = \deg(r) \geq 0$. By lemma 2 and the inductive hypothesis (using lemma1):

$$R_{n,m}(f, g) = a_n^{m-k} R_{n,k}(f, r) = a_n^{m-k} a_n^k \prod_{i=1}^{n} r(\alpha_i) = a_n^m \prod_{i=1}^{n} g(\alpha_i)$$

Case 1b: If $r = 0$ then $g = qf$ but $n > 0$ and $S_{n,m}(f, r) = S_{n,m}(f, 0)$ with the last n rows zero so $R_{n,m}(f, r) = 0$ gives $R_{n,m}(f, g) = 0$. Since $g(\alpha_1) = q(\alpha_1) f(\alpha_1) = 0$ so $\beta_k - \alpha_1 = 0$ and $R_2(f, g)$ vanishes as well.

Case 2: Suppose $n = 0$. This case has been shown true in the initial remark starting the induction.

Case 3: If $m < n$ with $\deg(g) = m$ and $\deg(f) = n$.
This reduces to case 1 and 2 by $R_{m,n}(g, f) = (-1)^{nm} R_{n,m}(f, g)$

Case 4: If $\deg(g) < m$ or $\deg(f) < n$ then the situation is handled by using lemma 2.
If both $\deg(g) < m$ and $\deg(f) < n$ then $a_n = b_m = 0$ gives $R_1(f, g) = R_2(f, g) = 0$.

This proof has been taken from a paper “Resultant and discriminant of polynomials”. It popped up when I made an Internet search and it just happened to be written by Svante Janson who was the team leader when I was taking part in the International Mathematical Olympiad in Paris in 1983.