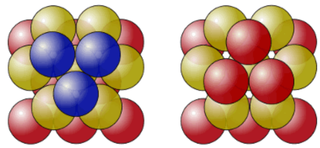
Appendix A

Exercises

Chapter 1. Introduction

1. Calculate the proportion of space that is occupied by spheres in the regular packing



2. Write a program that calculates the period and the different parts in the decimal expansion of a quotient $\frac{p}{q} = a_1 \dots a_i . b_1 \dots b_j \overline{c_1 \dots c_k}$.
   Check the statements made in the chapter.

Chapter 2. Origins

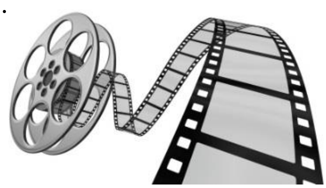
1. Which fractions have a finite sexagesimal expansion?
   Calculate the sexagesimal expansion of the first sexagesimally periodic fraction in the series $(1/n)_{n=1}^{\infty}$.

2. Pick a number $a_1 > 0$ and let $a_{k+1} = \frac{a_k + N/a_k}{2}$. Show that:
   $$\lim_{k\to\infty} a_k = \sqrt{N}$$
   This could be the method behind the approximation on YBC 7289.

3. Show that every fraction can be written as an Egyptian fraction:
   $$\frac{p}{q} = N + \sum_{k=1}^{n} \frac{1}{d_k} \quad p, q, N, n, d_k \in \mathbb{Z} \text{ and } 1 < d_1 < d_2 < \cdots < d_n$$

4. Show that every fraction can be written as an Egyptian fraction in an infinite number of ways.

5. Estimate an upper bound for the number of different books, images and movies that can be made.

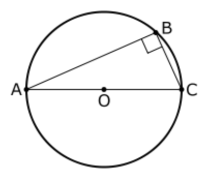


6. Three persons; one with 7 asava horses, one with 9 haya horses and one with 10 camels. Each gives two animals, one to each of the others. They are then equally well off. Find the value of each animal and the total value of the animals possessed by each person. All values are integers. (The problem occurs in the Bakshali manuscript)

7. Assume you can approximate $\sqrt{x}$, as in Mesopotamia ~1500 BC. Show a way to approximate $x^{p/q}$ where $x \in \mathbb{R}^+$ and $p, q \in \mathbb{Z}^+ (= \mathbb{N}_1)$

8. In Ramayana, a Sanskrit epic poem one of the characters Ravana sends two spies Shuka and Sarana to estimate the strength of the army of monkeys that builds the land bridge to Sri Lanka. According to Sarana their number is 100 crores of mahaughas $(= 10^2 \cdot 10^7 \cdot 10^{60})$.[8b] How reasonable is Sarana's estimation?

9. Prove Thales' theorem: If A, B and C are points on a circle with diameter AC then angle B is 90º



10. Show the inequalities $H \leq G \leq A$ among the Pythagorean means where $A = \frac{x+y}{2}$, $G = \sqrt{x \cdot y}$ and $H = \left(\frac{1/x+1/y}{2}\right)^{-1}$ where $x, y \in \mathbb{R}^+$.

11. Describe the three regular convex n-polytopes of each dimension ≥ 5.

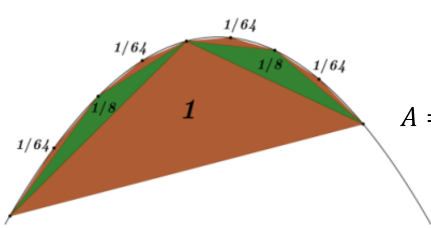
12 Express the fraction $\frac{100\,000}{101\,001}$ from chapter one as a continued fraction and show that $[1; 1,1, \ldots]$ equals the golden ratio $\varphi = (1 + \sqrt{5})/2$.

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

13. Derive the area of a disk by using a rectangular decomposition.

14. Do what Liu Hui failed to do, derive the volume of a sphere. (Without replicating page 64.)

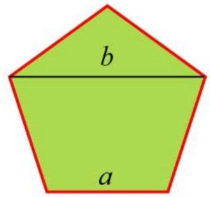
15. Show that the area of a parabolic segment can be seen as a sum of areas of inscribed triangles that form a geometric series.



$$A = A_1 \cdot \sum_{1}^{\infty} \frac{2^i}{8^i} = \frac{4}{3} A_1$$

16. Solve the cattle problem of Archimedes described on page 65.

17. Show that the ratio of the diagonal to the side in a regular pentagon equals the golden ratio, $\frac{b}{a} = \varphi \equiv \frac{1+\sqrt{5}}{2}$.



Chapter 3. Basics

1. Show that a logical n-ary operator $Q(P_1, \ldots, P_n)$ with a specified truth table can be given by a formula based on P_i , $\neg$, and $\wedge$.

	P_1	P_2	$\cdots$	P_{n-1}	P_n	Q	
$(2^n-1)_2$	1	1	$\cdots$	1	1	T_1	
$(2^n-2)_2$	1	1	$\cdots$	1	0	T_2	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$T_i \in \{0,1\}$
$(1)_2$	0	0	$\cdots$	0	1	T_{2^n-1}	
$(0)_2$	0	0	$\cdots$	0	0	T_{2^n}	

2. Conway's arrow notation $c_1 \to c_2 \to \cdots \to c_n$ is explained on page 91. Knuth's up-arrow notation $a \uparrow^b c$ is explained on page 29. Show that:
   - Conway chained arrow notation is not an iterated binary operator and
   - $p \to q \to r = p \uparrow^r q$
   - Express $3 \to 3 \to 3 \to 2$ in up-arrow notation.

3. Show that a sum of powers of degree p is a polynomial of degree $p+1$ and derive the polynomial $S_p(n)$ for $p = 3$, $p = 4$ and beyond.

$$S_p(n) = \sum_{k=1}^{n} k^p$$

4. Prove that if two sets are countable, totally ordered, dense and without upper and lower bounds then they are order-isomorphic.

5. Exercises on cardinality of sets:
   a) Show that $|\mathbb{R}| = |(0,1)|$.
   b) Show that $|\mathcal{P}(A)| > |A|$ for any set A.
   c) Show $|A| \le |B|$ and $|B| \le |A| \Rightarrow |A| = |B|$.
   d) Find a bijective function $f: [0,1] \to [0,1)$.

6. Prove the binomial identities from page 103.

7. Prove the multinomial theorem:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+\cdots+k_m=n} \binom{n}{k_1, k_2, \dots, k_m} {x_1}^{k_1} {x_2}^{k_2} \dots {x_m}^{k_m}$$

8. Use the definitions from page 104 of Stirling numbers $\left\{ n \atop k \right\}$ in terms of combinatorics or surjective functions to prove the algebraic identity:

$$\left\{ n \atop k \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

9. 

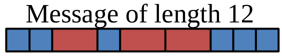
According to legend there is a temple with monks and 64 golden disks resting on three pillars. Ancient rules dictate that a disk may never rest on a smaller disk. When all disks have been moved the world will end. They are working day and night moving one disk every second. What is the shortest time to move all 64 golden disks?

10. How many different messages of length n can be built from symbols of length 1 and length 2?

$n = 1$, {  }

$n = 2$, {  }

Message of length 12



Compare the growth rate with a geometric sequence.

11. Prove Euler-Hierholzer's theorem from graph theory. A connected graph has an Euler cycle if and only if every node is of even degree.

12. Show that the set of numbers $\mathbb{Q}[\sqrt{2}] \coloneqq \{q_1 + q_2\sqrt{2} | q_1, q_2 \in \mathbb{Q}\}$ form a field (see page 97) under ordinary addition and multiplication.

13. Show equivalence of the different definitions of multiplicity k.

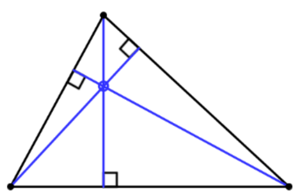
$$\begin{matrix} (z-\alpha)^k | P(z) \\ (z-\alpha)^{k+1} \nmid P(z) \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} P^{(i)}(\alpha) = 0 \text{ for } i \in \{0,1,\dots,k-1\} \\ P^{(k)}(\alpha) \neq 0 \end{matrix}$$

14. The location of a pirate treasure is described as follows:
Go from the gallows to the oak, turn 90 degrees to the left, walk the same distance and put a knife in the ground. Go back to the gallows, walk to the pine, turn 90 degrees to the right, walk the same distance and put another knife in the ground. On a line between the knives, dig half-way and you will find the treasure.
Descendants of the pirate found the description. They went to the island and found the pine and the oak but no gallows but still they could locate the treasure. Describe where they found it.

15. Show $e^z e^w = e^{z+w}$ for $z, w \in \mathbb{C}$.

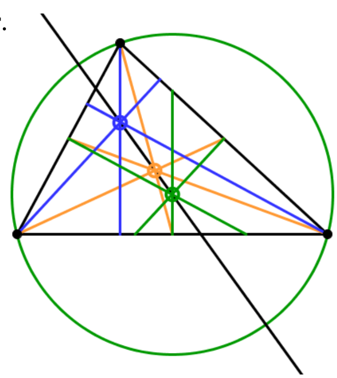
16. A Graeco-Latin square or an Euler square of order n is an arrangement of symbols from $\mathrm{G} = \{\alpha, \beta, \gamma, \ldots\}$ and $\mathrm{L} = \{a, b, c, \ldots\}$ with $|\mathrm{G}|=|\mathrm{L}|=n$ in such a way that each cell of an $n \times n$ square contains an ordered pair $(g, l) \in \mathrm{G} \times \mathrm{L}$. Every row and every column contain each element of G and each element of L exactly once and no cells contain the same pair. Euler presented the problem for $n = 6$ with $\mathrm{G} = \{\text{officer ranks}\}$ and $\mathrm{L} = \{\text{regiments}\}$, "the thirty-six officers' problem". He constructed Graeco-Latin squares for n=$2k + 1$ and n=$4k$ and conjectured that none exists for n=$4k + 2$. Show that he was wrong!

A similar problem with n=4 and 16 playing cards, $\mathrm{G} = \{\mathrm{A, K, Q, J}\}$ and $\mathrm{L} = \{♣, ♦, ♥, ♠\}$ has an extra constraint. Each diagonal should also contain all four face values and all four suits.
How many solutions are there?

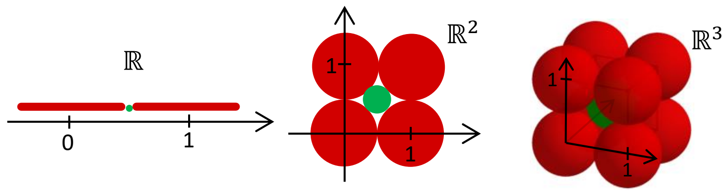


17. Show that the three altitudes of a triangle have one point in common, (the orthocenter).

18. Show that orthocenter, centroid and circumcenter of a non-equilateral triangle are collinear.



19. Explore how the radius varies with dimension of a sphere that is squeezed in between spheres centered at integer coordinates $\mathbb{Z}^n$ in $\mathbb{R}^n$.



20. Show that the two definitions of limit on page 165-166 are equivalent.

21. Show that if $\lim_{x \to c} f(x) = A$ and $\lim_{x \to c} g(x) = B$ then

$a)\ \lim_{x \to c}(f(x) \cdot g(x)) = A \cdot B$
$b)\ \lim_{x \to c}(f(x)/g(x)) = A/B \quad \text{if } B \neq 0$

23. Is there a function $f \in \mathrm{C}^0(\mathbb{R})$ such that f is continuous on $\mathbb{Q}$ but not on $\mathbb{R} \setminus \mathbb{Q}$?

24. Prove the Archimedean property for $\mathbb{R}$:
There is no positive real pair x, y such that $n \cdot x < y$ for every $n \in \mathbb{N}$.

[In an algebraic structure with such a pair.
x is said to be infinitesimal with respect to y and y is infinite w.r.t. x.
Elements that are infinitesimal in relation to the unit are infinitesimal.
Elements that are infinite in relation to the unit are infinite.]

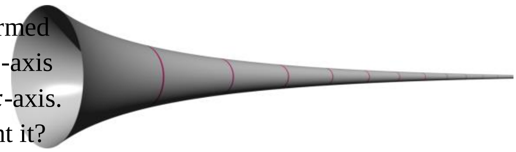
25. Prove that if f is continuous on a compact interval $[a, b]$ then f is uniformly continuous on that interval.

26. Assume that $f: [a, b] \to [c, d]$ is continuous and invertible and that f^{-1} is differentiable. Show that
$\int f^{-1} dy = y \cdot f^{-1}(y) - F \circ f^{-1}(y) + C$
Give the equation a figurative interpretation, a proof without words.

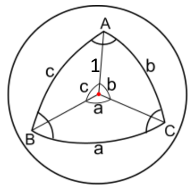
27. Show that the Cantor function also known as the Devil's staircase $c: [0,1] \to [0,1]$ which is described on page 192 is:
increasing, surjective, continuous and has a graph of arc length 2.

28. Show that the area $\int_\alpha^\beta f(x)\,dx$ for $f(x) = 1/x$ is unaffected by a rescaling of boundaries $[\alpha, \beta] \curvearrowright [c\alpha, c\beta]$.

29. Calculate the volume and area formed by rotating $y = 1/x$ around the x-axis for the interval $[1, \infty)$ along the x-axis. How much paint to fill it and paint it?

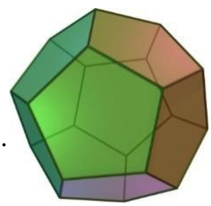


30. Show that the spherical law of cosines $\cos c = \cos a \cos b + \sin a \sin b \cos C$ reduces to the planar law of cosines $c^2 = a^2 + b^2 - 2ab \cos C$ as $a, b, c \to 0$.



31. A pyramid has an equilateral triangle as base, the sides are isosceles triangles and the height of the pyramid equals the distance between the height and the base. What is the angle between two sides?

32. Prove the spherical formula $\cos c = \cos a \cos b + \sin a \sin b \cos C$.



33. Calculate the inner angle between adjacent faces in a regular dodecahedron with regular pentagons as faces. (dodecahedron from Greek, do-2 deca-10 →12 faces)

34. Use the definition of hyperbolic functions from a hyperbola to show.

$\cosh A = \frac{1}{2}(e^A + e^{-A})$ $\qquad$ $\operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\sinh A = \frac{1}{2}(e^A - e^{-A})$ $\qquad$ $\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

35. Derive the Taylor series expansions of $\ln(x + 1)$, $\arctan x$ and $\operatorname{artanh} x$ around $x = 0$ and show:

$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

36. Calculate $f_\omega(3)$ and show that $f_{\omega^2}(n) > n \to \cdots \to n$ (n n's)

f_α comes from the fast-growing hierarchy.

$f_0(n) = n + 1$

$f_{\alpha+1}(n) = f_\alpha^n(n)$

$f_\alpha(n) = f_{\alpha_n}(n)$ when $\alpha = \lim_n \alpha_n$ is a limit ordinal.

37. A real or complex series $\sum_{k=0}^{\infty} a_k$ is said to be **absolutely convergent** if $S_n = \sum_{k=0}^{n} |a_k|$ is limited ($\sum_{k=0}^{\infty} |a_k| = \sup\{S_n | n \in \mathbb{N}_0\} = S$).
A series $\sum_{k=0}^{\infty} b_k$ that is **convergent** $\left(\lim_{n\to\infty} (\sum_{k=0}^{n} b_k) \in \mathbb{C}\right)$ without being absolutely convergent is **conditionally convergent**. Show that:

I. Absolute convergence $\Longrightarrow$ convergence.
II. Sum of absolutely convergent series is independent of the ordering order of the terms.
III. The sum of a real conditionally convergent series can attain any real number with an appropriate summation order.

38. Show that every solution to $\mathcal{L}(y) = y^{(n)} + a_{n-1}y^{n-1} + \cdots + a_0 y = 0$ with characteristic polynomial $l(r) = \prod_{k=1}^{\nu} (r - r_k)^{n_k}$ is of the form $y(x) = \sum_{k=1}^{\nu} P_k(x) e^{r_k x}$ with $\deg P_k < n_k$.

39. Homogeneous linear recurrence relation with constant coefficients of order n: $a_k = c_1 a_{k-1} + c_2 a_{k-2} + \cdots + c_n a_{k-n}$ $(*)$ $\quad a_i, c_i, r_i \in \mathbb{C}$
Characteristic polynomial $p(t) = t^n - \sum_{i=1}^{n} c_i t^{n-i} = \prod_{j=1}^{\nu} (t - r_j)^{n_j}$
Show that $a_k = \sum_{j=1}^{\nu} P_j(k) {r_j}^k$, P_j polynomials $\deg P_j < n_j$ solves $(*)$

40. The weighted power mean M_p of $x_1, \dots, x_n \in \mathbb{R}^+$ with weights $w_i \in \mathbb{R}^+$ and $\sum_{i=1}^{n} w_i = 1$ is defined by

$M_p(x_1, \dots, x_n) = (\sum_{i=1}^{n} w_i {x_i}^p)^{1/p}$ for $p \in \mathbb{R} \setminus \{0\}$
$M_0(x_1, \dots, x_n) = \prod_{i=1}^{n} {x_i}^{w_i}$
$M_{-\infty}(x_1, \dots, x_n) = \min(x_1, \dots, x_n)$
$M_{\infty}(x_1, \dots, x_n) = \max(x_1, \dots, x_n)$

$p = -1$	: harmonic mean
$p = 0$	: geometric mean
$p = 1$	: arithmetic mean
$p = 2$	: square mean

Show:
$\lim_{p\to 0} M_p = M_0$
$\lim_{p\to -\infty} M_p = M_{-\infty}$
$\lim_{p\to \infty} M_p = M_{\infty}$
$p < q \implies M_p(x_1, \dots, x_n) \le M_q(x_1, \dots, x_n)$
equality iff $x_1 = x_2 = \cdots = x_n$

($\min \le H.M \le G.M \le A.M \le S.M \le \max$)

Chapter 4. Return

1. Prove $p^2 | 2^{p(p-1)} - 1$ when p is a prime.

2. Show that if a fraction a/p with $0 < a < p$ and p a prime has a decimal expansion with even period $a/p = 0.\overline{r_1 \dots r_n r_{n+1} \dots r_{2n}}$ then $r_i + r_{i+n} = 9$

   Example: $\frac{1}{17} = 0.\overline{\underbrace{05882352}_{A}\underbrace{94117647}_{B}}$ $\quad \begin{array}{r} 05882352 \\ \underline{94117647} \\ 99999999 \end{array} \quad A + B = 10^n - 1$

3. There are infinitely many triples $(a, b, c) \in (\mathbb{Z}^+)^3$ with $\gcd(a, b, c) = 1$ and $a + b = c$ s.t. $q(a, b, c) > 1$.
   [$q(a,b,c) = \frac{\log(a,b,c)}{\log(\mathrm{rad(a,b,c)})}$ and $\mathrm{rad}\left(\prod p_i^{k_i}\right) = \prod p_i$]
   The ABC-conjecture states that for any $\varepsilon \in \mathbb{R}^+$
   there are only finitely many triples s.t. $q(a, b, c) > 1 + \varepsilon$.
   If the ABC-conjecture is true there is a maximal value of $q(a, b, c)$.
   Show that if $q(a, b, c) < 2$ then $a^n + b^n \neq c^n$ for $n \geq 6$.
   ABC says nothing about the limit of $q(a, b, c)$,biggest known is 1.63.

Chapter 5. History

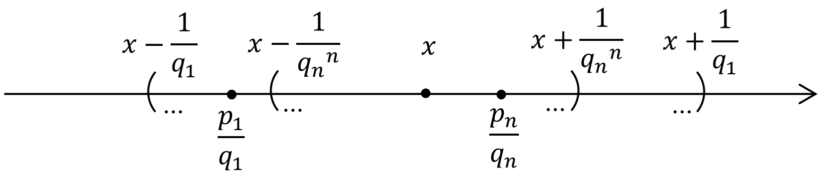
Chapter 6. Linear Algebra

1. Show that the inner angle between two adjacent faces
   in a regular dodecahedron equals $2\arctan(\varphi)$
   where $\varphi \equiv \left(\sqrt{5} + 1\right)/2$ is the golden ratio.
   Solve it by using a matrix for rotation.

Appendix C

1. Show that the Liouville numbers $\mathbb{L}$ are transcendental and that they form an uncountable dense subset of $\mathbb{R}$ with Lebesgue measure zero.

$$\mathbb{L} = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{Z}^+ \ \exists (p,q) \in (\mathbb{Z}, \mathbb{Z}^+ + 1) \left(\left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right) \right\}$$



2. Show that the Bernoulli numbers satisfy $B_{2k+1} = 0$ for $k \geq 1$.

3. Prove the properties of convolutions described in theorem 1 and 2.

$$\langle f_n \rangle \star \langle g_n \rangle = \langle \sum_{k=0}^{n} f_k g_{n-k} \rangle \text{ and } \langle f_n \rangle \star^b \langle g_n \rangle = \langle \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k} \rangle$$

are commutative and associative operators with identity $\langle 1,0,0, \dots \rangle$ and have a unique inverse for sequences $\langle a_0, a_1, a_2, \dots \rangle$ with $a_0 \neq 0$.

4. Use the formula for the resultant, $\Delta(P) = (-1)^{n(n-1)/2} R(P,P')/a_n$ of $P = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z - r_1)(z - r_2) \dots (z - r_n)$ with $R(P,Q) = |S_{P,Q}|$ where $S_{P,Q}$ is the Sylvester matrix to find the discriminant of $ax^4 + bx^3 + cx^2 + dx + e$ and check that the result is in accordance with the definition $\Delta(P) \equiv a_n^{2n-2} \cdot \prod_{1 \le i < j \le n} (r_i - r_j)^2$.

5. The resultant $R(f,g)$ of two polynomials with coefficients in a field $\mathbb{F}$, $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$ with roots $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_m$ in the algebraic closure of $\mathbb{F}$ can be defined in two alternate ways:

1. $$R(f,g) \equiv a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j)$$

2. $$R(f,g) \equiv \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \cdots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \cdots & b_2 & b_1 & b_0 \end{vmatrix}$$

Show that the two definitions are equivalent.